5th Olympiad of Metropolises

Mathematics · Day 1

Problem 1. In a triangle ABC with a right angle at C, the angle bisector AL (where L is on segment BC) intersects the altitude CH at point K. The bisector of angle BCH intersects segment AB at point M. Prove that $CK = ML$.

Problem 2. Does there exist a positive integer n such that all its digits (in the decimal system) are greater than 5, while all the digits of n^2 are less than 5?

Problem 3. Let $n > 1$ be a given integer. The Mint issues coins of n different values a_1, a_2, ..., a_n, where each a_i is a positive integer (the number of coins of each value is unlimited). A set of values $\{a_1, a_2, \dots, a_n\}$ is called *lucky*, if the sum $a_1 + a_2 + \cdots + a_n$ can be collected in a unique way (namely, by taking one coin of each value).

(a) Prove that there exists a lucky set of values $\{a_1, a_2, \dots, a_n\}$ with

$$a_1 + a_2 + \cdots + a_n < n \cdot 2^n.$$

(b) Prove that every lucky set of values $\{a_1, a_2, \dots, a_n\}$ satisfies

$$a_1 + a_2 + \cdots + a_n > n \cdot 2^{n-1}.$$