

5th Olympiad of Metropolises

Mathematics · Day 2

Problem 4. Positive numbers a , b and c satisfy $a^2 = b^2 + bc$ and $b^2 = c^2 + ac$. Prove that $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$.

Problem 5. There is an empty table with 2^{100} rows and 100 columns. Alice and Eva take turns filling the empty cells of the first row of the table, Alice plays first. In each move, Alice chooses an empty cell and puts a cross in it; Eva in each move chooses an empty cell and puts a zero. When no empty cells remain in the first row, the players move on to the second row, and so on (in each new row Alice plays first).

The game ends when all the rows are filled. Alice wants to make as many different rows in the table as possible, while Eva wants to make as few as possible. How many different rows will be there in the table if both follow their best strategies?

Problem 6. Consider a convex pentagon $ABCDE$. Let A_1, B_1, C_1, D_1, E_1 be the intersection points of the pairs of diagonals BD and CE , CE and DA , DA and EB , EB and AC , AC and BD , respectively. Prove that if four of the five quadrilaterals AB_1A_1B , BC_1B_1C , CD_1C_1D , DE_1D_1E , EA_1E_1A are cyclic, then the fifth one is also cyclic.