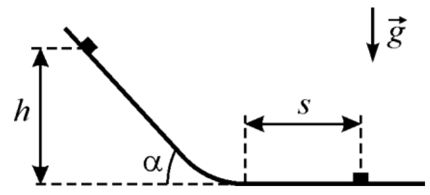
Problem 1: «So different friction»

Part 1: Impossible friction.

A dumb-bell made of two small massive spheres connected by a light rigid rod is held in vertical position so that the low (the lighter sphere) is on a horizontal rough surface. The masses of the spheres differ by the factor of 1.5. The dumb-bell is then carefully released and starts falling. There is no air drug. The low sphere begins to slide on the surface when the angle between the rod and the vertical becomes equal to $\alpha_1 = 30°$. Determine the coefficient of dry friction μ_1 between the low sphere and the surface. What must the friction coefficients (μ_2 and μ_3) be in order for the sliding to begin at the angles between the rod and the vertical equal to $\alpha_2 = 45°$ and $\alpha_3 = 60°$? All answers must be up to hundredths. Determine (in degrees, up to tenths) the largest possible angle of deflection from the vertical at which the low sphere can still start sliding (i.e. if the sphere did not start sliding before this angle had been reached, it would not slide until the upper sphere hit the surface).

Part 2: Friction on kink

Let us imagine an experiment when a small heavy puck is sliding downward a plastic chute from a height *h* with no initial velocity. The chute consists of three parts: a straight incline, a straight horizontal, and a junction connecting smoothly the first two. The junction is a cylindrical surface (see the Figure). The curvature radius of the surface is much less than *h*, much greater than the puck, and is independent of α. It turns out the puck remains at rest on the incline if $\alpha \le \alpha_c = 30°$. If the puck is released from some initial height h and $\alpha = 60°$, its braking distance on the horizontal part equals $s = 102$ cm. Determine the braking distance of the puck released from the height $h' = 2h$ at $\alpha' = 45°$ (the answer should be in centimeters). Estimate (in percent) the accuracy of the result if the curvature radius of the junction equals 7 cm. The coefficient of dry friction between the puck and the chute is assumed to be the same everywhere.

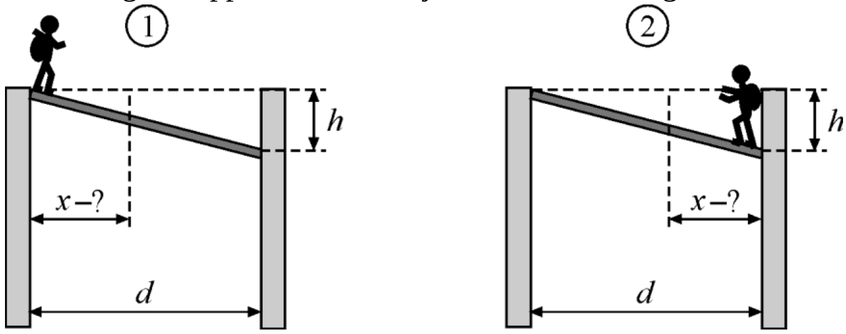


Part 3: Crossing «by friction».

A straight uniform log got stuck in a deep rocky crevice with parallel vertical walls and a width $d = 4$ m, the log is being held by friction. One butt of the log is at the upper edge of the crevice, the other one is lower by $h = 0,9$ m. The log lies in a vertical plane perpendicular to the crevice walls. The log mass equals $m = 80$ kg. The friction coefficient between the log butts and the walls equals $\mu = 0,9$. Consider two different situations.

1) A tourist 1 whose mass (the backpack included) is $M = 120$ kg, decided to cross the crevice by walking on the log. When he had stand on the upper end of the log, it slipped downward just a bit but the log held on.

2) A tourist 2, whose mass (the backpack included) is $M = 120$ kg, decided to cross the crevice by another log with the same parameters. When she carefully descended the crevice edge and stepped on the lower end of the log, it slipped downward just a bit but the log held on.

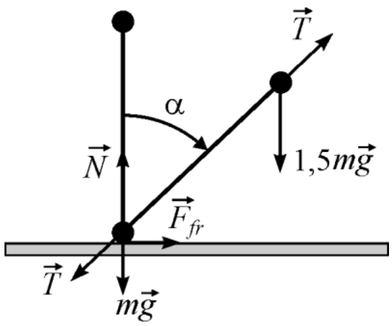


Let us assume that a log bend is negligible in both cases. Which tourist will be able to reach the end of the log (write down 1 or 2 as an answer)? For the tourist who should have not walked the log, determine the distance *x* (from the wall where he/she started) at the moment the log starts falling in the crevice. As an answer, write down the formula that includes only the quantities given in the problem statement and the numerical value in meters, up to hundredths.

In a more realistic model the log is allowed to bend. Consider the tourist crossing that was a success. Estimate the average radius of the log bend at the moment the tourist is precisely at the log midpoint (as an answer, write down the formula that includes only the quantities given in the problem statement and the numerical value in meters). How does the log bend affect the chances to fall (increases – 1, decreases – 2, does not affect – 0)? A wood density is $\rho = 630$ kg/m^3, a Young modulus for log compression along the grain is $E = 10^{10}$ Pa, the shear deformations are negligible, the log diameter is almost constant, and the free fall acceleration is $g \approx 10$ m/s^2.

Proposed solution:

1. Before the lower sphere starts sliding, the upper sphere follows a circle. The forces exerted on the upper sphere are the gravity force and the reaction force T by the rod. Let us write down the equation of motion for upper sphere in projection on the rod: $1{,}5m\dfrac{v^2}{L}=1{,}5mg\cos\alpha-T$ (here m is the mass of lighter sphere, v is a velocity of the lighter sphere, L is the rod length, and α is the angle between the rod and the vertical). According to the law of conservation of mechanical energy, $v^2=2gL\left(1-\cos\alpha\right)$.



Using these relations one can find the dependence of the rod reaction force and the inclination angle: $T(\alpha)=\dfrac{3}{2}mg\left(3\cos\alpha-2\right)$.

Forces being exerted on the lower sphere are: the gravity force, the rod reaction force, a normal force of reaction of the surface N, and a friction force F_{fr}. The condition of equilibrium of the lower sphere yields:

$$\begin{cases} N=mg+T\cos\alpha=\dfrac{mg}{2}\left(9\cos^2\alpha-6\cos\alpha+2\right), \\ F_{fr}=T\sin\alpha=\dfrac{3}{2}mg\sin\alpha\left(3\cos\alpha-2\right). \end{cases}.$$

The friction force prevents the sphere from sliding until $|F_{fr}|\le\mu N$. Therefore, the angle at which the sliding starts and the friction coefficient are related as

$$\mu=\frac{3\sin\alpha\left|3\cos\alpha-2\right|}{9\cos^2\alpha-6\cos\alpha+2}.$$

Substitution of the given angles into this formula yields the coefficients $\mu_1\approx0{,}252$, $\mu_2\approx0{,}114$, and $\mu_3\approx1{,}039$. It should be noted though, that the value at $\alpha_2=45°$ is impossible (it is less than at $\alpha_1=30°$)! Actually, for $\mu\approx0{,}114$ the sliding starts at an angle less than 30° (at $\alpha\approx11°$). It is all about the non-monotonic nature of the obtained dependence $\mu(\alpha)$. Introducing the function $f(x)=\dfrac{3\sqrt{1-x^2}\,|3x-2|}{9x^2-6x+2}$, one can find that the condition $f'_x(x)=0$ gives an equation $39x^2-40x+6=0$. Therefore, the function $\mu(\alpha)$ has maxima at $\alpha_{\mathrm{I}}=\arccos\left(\dfrac{20+\sqrt{166}}{39}\right)\approx32{,}5°$ and $\alpha_{\mathrm{II}}=\arccos\left(\dfrac{20-\sqrt{166}}{39}\right)\approx79{,}5°$. The function $\mu(\alpha)$ vanishes at $\alpha_0=\arccos\left(\dfrac{2}{3}\right)\approx48{,}2°$.

Therefore, there are values of α which are «impossible» for any μ and the angle $\alpha_2 = 45°$ belongs exactly to the domain of «impossible» values. Besides, we discovered that the maximum possible angle for the sliding to start is $\alpha_{\text{II}} = \arccos\left(\frac{20-\sqrt{166}}{39}\right) \approx 79{,}5°$: if the rod has tilted at this angle and the lower sphere did not start sliding, it will not slide until the upper sphere hits the surface.

2. Obviously, the coefficient of friction between the puck and the chute equals $\mu = \operatorname{tg}\alpha_c = \frac{1}{\sqrt{3}}$. Let r be the curvature radius of the junction. When the puck had slid down the incline, it gathered a speed v_1 determined by the law of conservation of mechanical energy:

$$\frac{mv_1^2}{2} = mg\left[h - r(1-\cos\alpha)\right] + A_{fr}.$$

The friction force on the incline is $|F_{fr}| = \mu mg\cos\alpha$, and the work it has done is negative ($A_{fr} = -|F_{fr}|\frac{h-r(1-\cos\alpha)}{\sin\alpha} = -\mu mg\operatorname{ctg}\alpha\left[h - r(1-\cos\alpha)\right]$). Taking into account that $r << h$, we obtain: $v_1^2 = 2g\left[h - r(1-\cos\alpha)\right](1-\mu\operatorname{ctg}\alpha) \approx 2gh(1-\mu\operatorname{ctg}\alpha)$. When analyzing the puck motion on the junction it is convenient to introduce a new variable – an angular coordinate φ counted from the beginning of the junction (this angle varies from 0 to α). The puck does not stop, therefore the friction force is a kinetic friction and equations for the tangential and centripetal accelerations are (N is the normal reaction force of the chute):

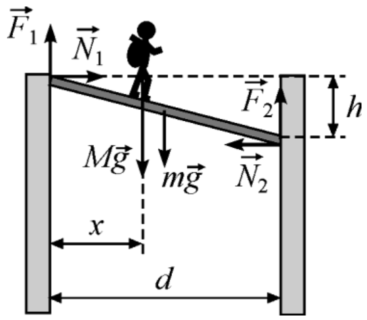
$$\left\{\begin{aligned} m\frac{dv}{dt} &= -\mu N + mg\sin(\alpha-\varphi) \\ m\frac{v^2}{r} &= N - mg\cos(\alpha-\varphi) \end{aligned}\right\} \Rightarrow \frac{dv}{dt} = -\mu\frac{v^2}{r} + g\left[\sin(\alpha-\varphi) - \mu\cos(\alpha-\varphi)\right].$$

(the angle between a tangent to the junction and the horizontal equals $\alpha - \varphi$). A small increment of the angle φ per a time increment dt is $d\varphi = \frac{v}{r}dt$, so $\frac{dv}{dt} = \frac{vdv}{rd\varphi} = \frac{1}{2r}\frac{d(v^2)}{d\varphi}$. Therefore, $\frac{d(v^2)}{d\varphi} = -2\mu v^2 + 2gr\left[\sin(\alpha-\varphi) - \mu\cos(\alpha-\varphi)\right]$. Again, we neglect terms of the order of gr and obtain $\frac{d(v^2)}{d\varphi} \approx -2\mu v^2$. The function which derivative is proportional to itself is exponent, so $v^2(\varphi) \approx const\cdot e^{-2\mu\varphi}$. Since $v^2(0) = v_1^2$, then $v^2(\varphi) \approx v_1^2\cdot e^{-2\mu\varphi}$. Thus, the puck velocity at the beginning of the horizontal satisfies to the relation $v_2^2 \approx v_1^2\cdot e^{-2\mu\alpha}$. Hence, up to corrections of the order of gr, the square of the velocity at the beginning of the horizontal is $v_2^2 \approx 2gh(1-\mu\operatorname{ctg}\alpha)\cdot e^{-2\mu\alpha}$. The braking distance on the horizontal part, where the puck acceleration is $|\vec{a}_s| = \mu g$, equals: $s = \frac{v_2^2}{2\mu g} \approx h\left[\frac{1}{\mu} - \operatorname{ctg}\alpha\right]\cdot e^{-2\mu\alpha}$. For the values given we obtain:

$s \approx \frac{2h}{\sqrt{3}} \cdot \exp\left(-\frac{2\pi}{3\sqrt{3}}\right)$ and $s' \approx 2h(\sqrt{3}-1)\exp\left(-\frac{\pi}{2\sqrt{3}}\right)$. Therefore, $s' \approx s(3-\sqrt{3})\exp\left(-\frac{\pi}{6\sqrt{3}}\right) \approx 175$ cm.

The terms discarded from the formulae for v_1^2 and v_2^2 are of the order of $2gr$. Thus, discarding these terms contributes a relative error of the order of $\frac{r}{h} \approx 0,024$, i.e. about 3 %.

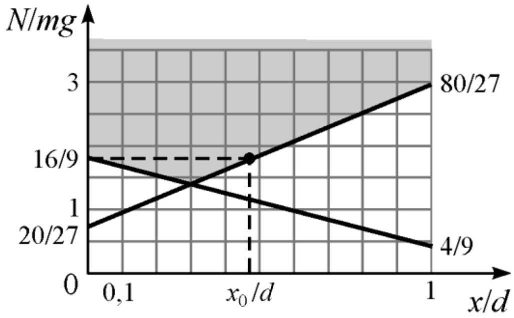
3. Let us examine the conditions for both butts of the log to do not slip at the moment the tourist 1 is at a distance x from the «left» edge of the crevice. Let us write down the conditions of the log equilibrium at this moment.



Balancing horizontal components of the forces yields the condition of equality of the forces of normal reaction: $N_1 = N_2 \equiv N$. The balance of the vertical components of the forces means that the sum of the friction forces exerted on the log butts balances the sum of the gravity forces: $F_1 + F_2 = (M+m)g$. Balancing the torques evaluated with respect to the upper end of the log gives the equation: $F_2 d - Nh - mg\frac{d}{2} - Mgx = 0$, which results in $F_2 = \frac{1}{2}mg + Mg\frac{x}{d} + N\frac{h}{d}$. Similarly, $F_1 = \frac{1}{2}mg + Mg\frac{d-x}{d} - N\frac{h}{d}$. The log butts do not slip if $F_{1,2} \le \mu N$. Thus, the log does not fall if the forces of normal reaction of the crevice walls balance the horizontal component of the log elastic force (due to log compression when being «pushed» in the crevice) satisfies:

$$\begin{cases} N \ge \dfrac{(2M+m)d - 2Mx}{2(\mu d + h)} g, \\ N \ge \dfrac{md + 2Mx}{2(\mu d - h)} g. \end{cases}$$

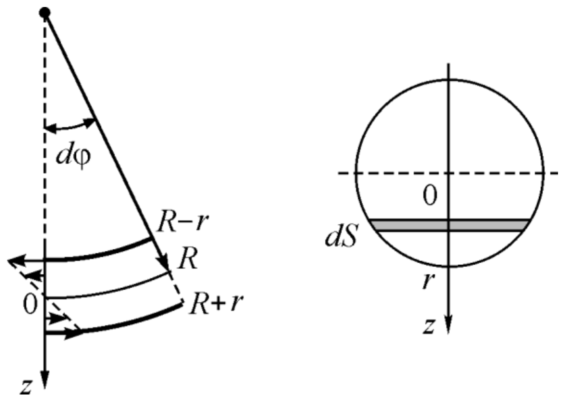
On the diagram below one can see the region (shaded grey) of the values of N, for which the log does not slip, as a function of x.



The value of N corresponding to the first situation is determined by the condition that the upper log butt at $x=0$ «slipped down just a little bit» but held on. Thus, $F_1=\mu N$ at $x=0$, i.e. $N=\dfrac{(2M+m)dg}{2(\mu d+h)}$. It is evident that for this value of the normal force a further increase of x will inevitably violate the condition of not sliding at the lower end and this will happen when $\dfrac{md+2Mx_0}{\mu d-h}=\dfrac{(2M+m)d}{\mu d+h}$, i.e. at $x_0=\left[(2M+m)\dfrac{\mu d-h}{\mu d+h}-m\right]\dfrac{d}{2M}\approx 1{,}87$ m. Thus, the tourist will fall into the crevice together with the log: if he dares to walk on it, he will even not reach the log midpoint!

In the situation 2, the value of N is determined by the condition $F_2=\mu N$ at $x=d$, and $N=\dfrac{(2M+m)dg}{2(\mu d-h)}$. Now, according to the diagram, the sliding does not begin at any x, so the tourist 2 will safely reach the opposite edge of the crevice.

To estimate the average curvature radius of the log axis let us assume that this radius is much larger than the log length $L\approx\sqrt{d^2+h^2}=4{,}1$ m and is constant at any point of the log. An equilibrium bend corresponds to the situation when the torque of elastic force balances the torque of external bending force in any log cross section. As long as the log remains at rest the points where it touches the walls can be regarded as the log «fixed» points and attribute the log bend to the action of gravity force exerted on the log and the tourist. To do an estimate, let us assume that the gravity force is applied to the log center, so the torque due to these forces with respect to a «fixed» point equals $\mathrm{M}_{ex}=(M+m)g\dfrac{d}{2}$. Let the log radius be r. Consider a cross section of the log which midline has bent with a curvature radius R with its length maintained. The log layers closer to the center of curvature than this line will be compressed and those which are further will be stretched.



Let us introduce a layer coordinate $-r\le z\le +r$ with respect to the midline (see the Figure). The log element of a length dl in the non-deformed state will have an angular size $d\varphi=\dfrac{dl}{R}$. Therefore, the deformation of this element in a layer of coordinate z equals $\delta l=(R+z)d\varphi-dl=\dfrac{z}{R}dl$. The elastic coefficient of the layer is $k=E\dfrac{dS}{dl}$ (dS is the layer cross section area), so the elastic force equals $dF_{el}=k\delta l=E\dfrac{z}{R}dS$ and its contribution to the total torque in a given cross section is $d\mathrm{M}_{el} = zdF_{el} = \frac{E}{R} z^2 dS$. Thus, the total torque of elastic force is $\mathrm{M}_{el} = \frac{E}{R}\int_{-r}^{+r} z^2 dS = \frac{4E}{R}\int_{0}^{+r} z^2\sqrt{r^2 - z^2}dz$. Evaluating the integral (e.g. by using the substitution $z = r\cdot\sin\alpha$: $\int_{0}^{+r} z^2\sqrt{r^2 - z^2}dz = r^4\int_{0}^{\pi/2}\sin^2\alpha\cos^2\alpha d\alpha = \frac{\pi r^4}{16}$), we obtain: $\mathrm{M}_{el} = \frac{\pi E r^4}{4R} = \frac{Em^2}{4\pi R\rho^2 L^2}$.

Here, we used that $\pi r^2 = \frac{m}{\rho L}$. Therefore, the equation $\mathrm{M}_{ex} = \mathrm{M}_{el}$ results in a relation $(M+m)g\frac{d}{2} = \frac{Em^2}{4\pi R\rho^2 L^2}$, which gives $R = \frac{Em^2}{2\pi(M+m)g\rho^2 d(d^2+h^2)} \approx 190$ m. One can see that the assumption $L << R$ is good enough. However, it is clear that the bend is actually not uniform and the gravity force is distributed over the log, so our calculations are indeed just an estimate.

The bend increases the length of the log axis (relative to the unloaded log), i.e. decreases its longitudinal deformation. Therefore, it decreases the force the compressed log exerts on the crevice walls and, accordingly, the normal reaction force of the walls. As it is clear from the diagram this increases the danger of falling (it turns out, quite essentially, so the second tourist should not walk the log as well).

Note. An approximate calculation admits simplifications which do not significantly alter the result. For instance, the log can be replaced with a beam of square cross section of the same area $S = \frac{m}{\rho L} \equiv a^2$. The integral becomes much simpler to evaluate: $\int_{-a/2}^{+a/2} z^2 dS = 2a\int_{0}^{+a/2} z^2 dz = \frac{a^4}{12}$, but the result is almost the same: $R = \frac{Em^2}{6(M+m)g\rho^2 d(d^2+h^2)} \approx 200$ m. It is also possible to estimate the torque by replacing the continuously distributed elastic forces by a pair of «average» forces with an absolute value $F_{el} = k\frac{r}{2R}dl = E\frac{\pi r^2}{2\cdot dl}\frac{r}{2R}dl = E\frac{\pi r^3}{4R}$ and the distance between the moment arms equal to r. Then the torque $\mathrm{M}_{el} = E\frac{\pi r^4}{4R}$ turns out to be the same as in the above calculation! Other reasonable estimates are also possible.

Problem 2: «To the Sun!»

The spacecraft launched into a low circular orbit at an altitude of h = 300 km above the Earth's surface had to perform two short-term acceleration maneuvers. As a result of these maneuvers the ship was supposed to be in a geostationary orbit (GSO) - this is a circular orbit lying in the plane of the Earth's equator, in which the object is always above the same point on the Earth's surface.

The first maneuver was successfully accomplished and the ship reached an intermediate elliptical orbit with the maximum distance between the ship and the Earth's center equal to the GSO radius while the minimum distance to the Earth's center remained equal to the radius of the initial orbit.

The second maneuver was designed so that the rocket engine would turn on only once and for a short time and use up one third of the fuel and oxidizer remained after the first maneuver. However, the engine did not start at the right moment. Although the technical crew was able to determine the cause of failure and regain control of the ship, the latter could not manage to get to the desired point at the GSO and its mission failed.

Then they decided to try another maneuver: sending a part of the spacecraft from the intermediate orbit to an orbit heading to the Sun. Before this maneuver the ship had to be split in two parts in such a way that a part with the engine went to the Sun and the second (as unnecessary) part was left in space. It turned out, the orbit parameters allowed one to choose a moment for the engine to start so that the part sent to the Sun had the maximum possible mass.

The following facts are considered known.

1. In polar coordinates the equation of motion of a body moving in the gravitational field of a massive spherical object is $\rho(\varphi) = \dfrac{p}{1 + \varepsilon \cdot \cos\varphi}$ (the origin is placed at the object center). Here p is the orbit parameter and ε is its eccentricity.

2. The distance between Earth and the Sun varies between $r_A \approx 152{,}1$ Mkm (at aphelion passing in July) and $r_P \approx 147{,}1$ Mkm (at perihelion passing in January). The Solar radius equals $R_S = 0{,}70$ Mkm. The period of Earth revolution around the Sun is $T_0 \approx 3{,}156 \cdot 10^7$ s. Both Earth and the Sun can be regarded as spherical bodies.

3. The Earth's radius is $R_E \approx 6371$ km. The duration of Earth's day is $T = 24 \times 3600$ c = 86400 s. The free fall acceleration at the Earth's surface equals $g \approx 9{,}807$ m/s^2. Rotation of Earth and the Sun around their axes is in the same direction. The ship discussed in this problem was launched in the direction of Earth's rotation as the majority of spacecraft.

4. The exhaust velocity of combustion products from the engine nozzle is $u = 6$ km/s (relative to the ship). According to *rocket equation*, when rocket velocity increases by Δv the rocket mass decreases from m to $m' = m \cdot e^{-\Delta v / u}$.

Please, answer the following questions.

3) Calculate the parameter p and the eccentricity ε of the intermediate orbit (write down the expressions for p and ε in terms of the quantities given in the problem and compute their numerical values with at least 1% accuracy). How much does the absolute value of the ship total mechanical energy differ for the original circular orbit and the intermediate orbit? To answer this question, calculate the numerical value of the ratio $|E_0 / E_1|$ (up to hundredths). To define potential energy of gravitational interaction between the ship and Earth use the standard assumption that the potential energy vanishes at infinite separation between them.

4) Think at which short section of the intermediate orbit the engine should had been turned on to accomplish the originally planned maneuver of transition from the intermediate orbit to GSO (you are not required to answer this question). What velocity increment Δv_2 would be achieved as a result of the maneuver (in km/s, up to hundredths)? What would the ship mass be after the maneuver (the answer should be given as the percentage of the original ship mass on the circular orbit with an accuracy of 1%).

Note: when calculating the numerical values to answer the questions below, keep no less than four significant digits in the intermediate results.

5) In which season (in the northern hemisphere) and at which approximate time of day should the engine be started in order to send the maximum (by mass) part of the ship to the Sun? An answer to this question must contain the season (**winter / spring / summer / autumn**) and the time of day (**sunrise / noon / sunset / midnight**). What is a required velocity v_A of the ship part heading toward the Sun in the Copernicus frame (in which the Sun is at rest) at the moment when the distance between the ship and Earth sufficiently exceeds R_E and is much less than r_A? What is the velocity $\tilde{v}_2$ of this part of the ship relative to Earth right after the engine is turned off? Write down both velocities in km/s up to hundredths.

6) What part of the ship mass (in percentage of its mass on the original circular orbit with an accuracy of 1%) should be detached as “unnecessary” in order to perform the maneuver? What is the mass (in percentage of the ship mass on the original circular orbit with an accuracy of 1%) of the part that will arrive to the Sun?

7) In what time after turning off the engine will the accelerated ship part reach the Sun? Write down the formula for the time in terms of the quantities given in the problem and calculate the numerical value (in Earth’s days, up to tenths).

8) “Gravity assist maneuver” is a change of spacecraft velocity caused by its passing close to a very massive celestial object. Suppose we are going to perform such a maneuver near Venus. If a ship is set to an orbit which perihelion is close to the perihelion of Venus orbit and both the ship and Venus arrive to their perihelia simultaneously, the correct maneuver will send the ship passing close to Venus toward the Sun. Evaluate the maximum possible angle the ship velocity vector can be turned as a result of the gravity assist maneuver near Venus. The answer should be given in degrees with an accuracy of one degree. The distance between Venus and the Sun in perihelion equals $r'_P \approx 107{,}5$ Mkm, the velocity of Venus at the perihelion is $V'_P \approx 35{,}25$ km/s, and the first cosmic velocity (the velocity on the low circular orbit near the surface) is $v_{1V} \approx 7{,}33$ km/s. Venus can be regarded as a spherical body. Is it possible to deliver to the

Sun a substantially greater part of the ship by using this maneuver rather than using the maneuver described in **3** and **4** (respond “yes” or “no”)?

Problem 2 (proposed solution)

1. Let us determine the radius of GSO and the ship velocity in this orbit (r and v, respectively). To do this we use the equation for the centripetal component of the ship acceleration in the GSO: $m\frac{v^2}{r} = \frac{GM_E m}{r^2} = \frac{gR_E^2 m}{r^2}$, hence $v = R_E\sqrt{\frac{g}{r}}$. On the other hand, the orbital period of the ship must be the same as the period of Earth's rotation around its axis, i.e. $\frac{2\pi r}{v} = T$, therefore $r = \sqrt[3]{\frac{gT^2 R_E^2}{4\pi^2}} \approx 42222$ km and $v = \sqrt[3]{\frac{2\pi g R_E^2}{T}} \approx 3{,}0705$ km/s. According to the orbit equation, one has for the intermediate orbit $R_E + h = \frac{p}{1+\varepsilon}$ and $r = \frac{p}{1-\varepsilon}$. These equations yield: $p = \frac{2r(R_E+h)}{r+R_E+h} \approx 11522$ km and $\varepsilon = \frac{r-R_E-h}{r+R_E+h} \approx 0{,}727$.

Below the symbols *a* and *p* stand for the quantities related to *aphelion* and *perihelion* of a ship orbit around Earth and symbols *A* and *P* relate to *aphelion* and *perihelion* of an orbit around the Sun. For an elliptical orbit of the ship both the total mechanical energy and angular momentum are conserved and can be expressed via the given distances from the Earth's center to the *aphelion* and the *perihelion*. Therefore, $\frac{mv_a^2}{2} - \frac{GM_E m}{r_a} = \frac{mv_p^2}{2} - \frac{GM_E m}{r_p} = E$ and $mr_a v_a = mr_p v_p$. Solving these equations for velocities, one obtains $v_p^2 = gR_E^2\frac{2r_a}{r_p(r_a+r_p)}$ and $v_a^2 = gR_E^2\frac{2r_p}{r_a(r_a+r_p)}$. This gives for the total mechanical energy: $E = -\frac{gR_E^2 m}{r_p + r_a}$. Using this result, one finds that $\left|\frac{E_0}{E_1}\right| = \frac{r+R_E+h}{2(R_E+h)}\frac{m_0}{m_1}$, where m_0 and m_1 are the ship masses in the initial and the intermediate orbits.

Since the radius of initial orbit equals the distance to the perihelion of intermediate orbit, the transition was performed by increasing the ship velocity from $v_0 = \sqrt{\frac{gR_E^2}{R_E+h}} \approx 7{,}7247$ km/s to $v_p = \sqrt{\frac{2gR_E^2 r}{(R_E+h)(r+R_E+h)}} \approx 10{,}152$ km/s for a short period of time (according to the problem statement the maneuver was «short-term»). Therefore, it is reasonable to neglect both the curvature of ship trajectory and a gravity force and to consider the ship acceleration as being completely due to the engine thrust. Hence, one can use the Tsiolkovsky rocket equation: $m_1 = m_0 \cdot e^{-\Delta v_1/u}$, where $\Delta v_1 \equiv v_p - v_0 \approx 2{,}4271$ km/s. Thus, $m_1 = 0{,}66730 \cdot m_0$ and $\left|\frac{E_0}{E_1}\right| = \frac{r+R_E+h}{2(R_E+h)}e^{\Delta v_1/u} \approx 5{,}49$. Note that both energies are negative, i.e. actually $E_0 < E_1$.

2. All ship orbits under consideration (with its engine turned off) near Earth are closed. Therefore, the short-term acceleration for the transition from the intermediate orbit into the GSO must be done immediately before the aphelion of intermediate orbit. To do this, it suffices to increase the ship velocity from v_a at the aphelion to the velocity in the GSO (it is important that both at the aphelion of intermediate orbit and in the GSO the velocity is perpendicular to the radius). Note that $v_a = \sqrt{\dfrac{2gR_E^2(R_E+h)}{r(r+R_E+h)}} \approx 1{,}6040$ km/s. Therefore, the velocity increment required to switch into the GSO must be $\Delta v_2 \equiv v - v_a \approx 1{,}4665$ km/s. Again, use the rocket equation: $m_2 = m_1 \cdot e^{-\Delta v_2/u} = m_0 \cdot e^{-(\Delta v_1 + \Delta v_2)/u} \approx 0{,}52260 \cdot m_0$. The mass increment $m_1 - m_2 \approx 0{,}14470 \cdot m_0$ is the mass of the fuel and oxidizer which has to be consumed to switch into the GSO from the intermediate orbit. According to the problem, it was one third of the total supply. Hence, the mass of fuel and oxidizer in the intermediate orbit was $m_F = 0{,}43410 \cdot m_0$. Since the ship mass in the intermediate orbit was $m_1 = 0{,}66730 \cdot m_0$, the «useful» mass (the engine, the fuel tanks, and the hull) was equal to $m_u = 0{,}23321 \cdot m_0$.

3. In order to send the ship to the Sun it is necessary to reduce significantly the ship velocity with respect to the Sun. To do this in a most efficient way, the engine must be started when the ship is closest to the point where its velocity in the intermediate orbit is minimal with respect to the Sun. Obviously, it would be the best to choose the point at which the Earth's velocity V with respect to the Sun is minimum (i.e. the aphelion of the Earth's orbit) and the ship velocity relative to Earth is maximum and opposite to $\vec{V}_A$. Therefore, the second maneuver must be performed when the ship approaches the perihelion of its orbit during the summer of the northern hemisphere, i.e. when Earth is at the aphelion, and at noon of the Earth's time at the point below the ship.

In order to reach the Sun, the ship after leaving Earth must follow an orbit around the Sun for which the distance to the aphelion is approximately equal to the distance to aphelion of the Earth's orbit and the distance to perihelion is less or equal to the Solar radius. Then the ship velocity relative to the Sun after the it has receded from Earth to a distance sufficiently greater than the Earths' radius (though much less than $r_A \approx 152$ Mkm) must not exceed $v_A = \sqrt{\dfrac{2GM_S R_S}{r_A(r_A+R_S)}}$.

The Solar mass can be estimated by using the orbital period of Earth and the mean radius of the Earth's orbit: $V = \dfrac{\pi(r_A+r_P)}{T_0} \approx \sqrt{\dfrac{2GM_S}{r_A+r_P}} \Rightarrow 2GM_S \approx \dfrac{\pi^2(r_A+r_P)^3}{T_0^2}$. Thus, the required velocity is $v_A \approx \dfrac{\pi(r_A+r_P)}{T_0}\sqrt{\dfrac{(r_A+r_P)R_S}{r_A(r_A+R_S)}} \approx 2{,}8273$ km/s. The Earth's velocity at the aphelion is $V_A = \sqrt{\dfrac{2GM_S r_P}{r_A(r_A+r_P)}} \approx \dfrac{\pi(r_A+r_P)}{T_0}\sqrt{\dfrac{r_P}{r_A}} \approx 29{,}2898$ km/s, so the ship velocity relative to Earth at the moment of its passage into the orbit leading to the Sun must be $v'_A = V_A - v_A \approx 26{,}4625$ km/s. It is important to keep in mind the following. After turning off the engine, the ship must move somewhat faster than v'_A to overcome the gravitational pull of receding Earth. This effect can be accounted for by means of the energy conservation law: the velocity $\tilde{v}_2$ of the part of the ship (of the mass m_2) being sent to the Sun relative to the Earth's center at the moment right after the engine was turned off is determined by the equation $\frac{m_2\tilde{v}_2^2}{2} - m_2 g R_E = \frac{m_2 (v'_A)^2}{2} \Rightarrow \tilde{v}_2 = \sqrt{(v'_A)^2 + 2gR_E} \approx 28{,}7267$ km/s.

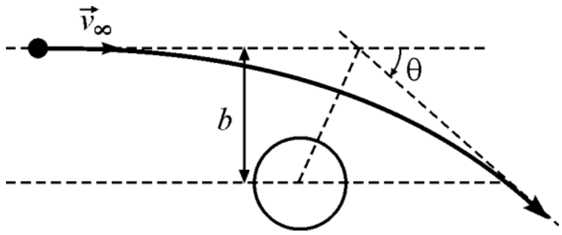
4. During the maneuver of transition into the orbit leading to the Sun the increment of the ship velocity is $\Delta\tilde{v}_2 \equiv v_2 - v_p \approx 18{,}575$ km/s. Before the acceleration the ship got rid of some of its mass (Δm), so the mass which will eventually reach the Sun is $\tilde{m}_2 = (m_1 - \Delta m)\cdot e^{-\Delta\tilde{v}_2/u}$. To send the maximum mass to the Sun, the ship had to use up all the fuel and oxidizer supply, therefore $m_F = m_1 - \Delta m - \tilde{m}_2 = (m_1 - \Delta m)\cdot(1 - e^{-\Delta\tilde{v}_2/u}) \Rightarrow \Delta m = m_1 - \frac{m_F}{1 - e^{-\Delta\tilde{v}_2/u}} \approx 0{,}21264\cdot m_0$. This calculation shows that the mass of the ship left after the second maneuver is $\tilde{m}_2 = \frac{m_F}{1 - e^{-\Delta\tilde{v}_2/u}} e^{-\Delta\tilde{v}_2/u} = \frac{m_F}{e^{\Delta\tilde{v}_2/u} - 1} \approx 0{,}02\cdot m_0$. Thus, no more than 2% of the ship mass in the initial orbit can be delivered to the Sun!

5. According to the third Kepler's law, a ratio of the squares of orbital periods equals to the cube of the ratio of the semi-major axes of the orbits. Therefore, the orbital period of the accelerated part of the ship in the new orbit is $T' = \left(\frac{r_A + R_s}{r_A + r_P}\right)^{3/2} T_0 \approx 133{,}3$ days. The trip to the Sun will take a half of this period, i.e. $t = \frac{1}{2}\left(\frac{r_A + R_s}{r_A + r_P}\right)^{3/2} T_0 \approx 66{,}7$ days.

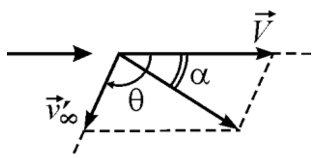
6. Detailed analysis of the gravity assisted maneuver requires a more general study of trajectories of a body moving in the gravitational field of a massive spherically uniform object. Let us again turn to the laws of conservation of mechanical energy and angular momentum. A ship following the orbit $\rho(\varphi) = \frac{p}{1 + \varepsilon\cdot\cos\varphi}$ has the total mechanical energy $E = \frac{m\vec{v}^2}{2} - \frac{GMm}{r} = \frac{m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2)}{2} - \frac{GMm}{\rho} = const$ and the angular momentum $L = m\rho^2\dot{\varphi} = const$, where the dot means differentiation with respect to time (the polar coordinates are employed). From the second relation one gets $\dot{\varphi} = \frac{L}{m\rho^2}$ and substituting into the first equation obtains: $\frac{E}{m} = \frac{\dot{\rho}^2}{2} + \frac{L^2}{2m^2\rho^2} - \frac{GM}{\rho}$. The distances to aphelion and perihelion, $r_a = \frac{p}{1-\varepsilon}$ and $r_p = \frac{p}{1+\varepsilon}$, are the roots of the equation $\dot{\rho} = 0$. Therefore (M_V is the Venus's mass):

$$\left\{\begin{aligned}\frac{p}{1-\varepsilon}&=-\frac{GM_V m}{2E}-\sqrt{\left(\frac{GM_V m}{2E}\right)^2+\frac{L^2}{2mE}}\\ \frac{p}{1+\varepsilon}&=-\frac{GM_V m}{2E}+\sqrt{\left(\frac{GM_V m}{2E}\right)^2+\frac{L^2}{2mE}}\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}p&=\frac{L^2}{GM_V m^2}\\ \varepsilon&=\sqrt{1+\frac{2L^2E}{m^3(GM_V)^2}}\end{aligned}\right..$$

When the part of the ship approaches Venus in the region of the orbits' closing, in the Venus's frame it comes to Venus from «a large distance» at non-zero velocity v_∞, i.e. its total mechanical energy $E=\frac{mv_\infty^2}{2}>0$ and the orbit eccentricity $\varepsilon>1$. In this case the orbit is hyperbolic, one can see that at $\varphi=\pm\left[\frac{\pi}{2}+\arcsin\left(\frac{1}{\varepsilon}\right)\right]$ the radius $\rho\to\infty$. Thus, the velocity relative to Venus rotates at the angle $\theta=2\arcsin\left(\frac{1}{\varepsilon}\right)$. In this case, $L=mv_\infty b$ where b (see the Figure) is called the «impact parameter».



One can see that the maximum rotation angle is achieved at the minimum eccentricity. According to the formula for eccentricity, $\varepsilon=\sqrt{1+\frac{b^2v_\infty^4}{(GM_V)^2}}$, and the minimum value of the impact parameter is determined by the requirement that the distance between the orbit perihelion and the Venus's center cannot be less than the planet radius R_V: $\frac{p}{1+\varepsilon}\geq R_V\Rightarrow b^2\geq\left(1+2\frac{GM}{R_V v_\infty^2}\right)R_V^2=\left(1+2\frac{v_{1V}^2}{v_\infty^2}\right)R_V^2$. Thus, $\varepsilon_{\min}=1+\frac{v_\infty^2}{v_{1V}^2}$, and the maximum rotation angle achieved during the flyby around Venus is $\theta_{\max}=2\arcsin\left[\frac{v_{1V}^2}{v_{1V}^2+v_\infty^2}\right]$. The ship velocity relative to the Sun after the gravity assist maneuver is a vector sum of velocity $\vec{v}'_\infty$ rotated by the angle θ and the velocity $\vec{V}$ of the massive object relative to the Sun. Thus, the maximum angle $\alpha_{\max}$ the ship velocity vector can rotate toward the Sun during the maneuver can be found by means of the law of sines: $\sin\alpha=\frac{v_\infty}{|\vec{V}+\vec{v}'_\infty|}\sin\theta$. Finally,



$$\alpha_{\max}=\arcsin\left(\frac{v_\infty\sin\theta_{\max}}{\sqrt{V^2+v_\infty^2+2Vv_\infty\cos\theta_{\max}}}\right).$$

Let us determine the velocity of the part of ship at the perihelion of its orbit: $v_P = \sqrt{\dfrac{2GM_S r_A}{r'_P(r_A + r'_P)}}$,

or $v_P = \dfrac{\pi(r_A + r_P)}{T_0}\sqrt{\dfrac{r_A(r_A + r_P)}{r'_P(r_A + r'_P)}} \approx 38{,}0332$ km/s (recall that the aphelion of the orbit leading from Earth to Venus coincides with the aphelion of the Earth's orbit and the perihelion with the perihelion of the Venus's orbit.). So, $v_\infty = v_P - V'_P \approx 2{,}7832$ km/s, $V = V'_P \approx 35{,}25$ km/s, and $\theta_{\max} = 2\arcsin\left[\dfrac{v_{1V}^2}{v_{1V}^2 + v_\infty^2}\right] \approx 121{,}9°$. Therefore $\alpha_{\max} \approx 4°$. However, to hit the Sun at this velocity the angle rotation must be close to 90°. Thus, the gravity assisted maneuver near Venus cannot help in delivering a substantially larger ship mass to the Sun.

Problem 3: «PLANETARY NEBULA»

Planetary nebula is a cloud of gas bound by gravity of its «core» (typically a massive star). If the cloud is large enough we do not see the star itself because the light emitted by the star undergoes multiple scattering by the gas particles, is absorbed by them, and then reemitted. Thus, in our telescopes we observe only a shining nebula. In this problem you are suggested to study a relation between the nebula radiation seen by a distant observer and the original radiation of the core star. In reality such a nebula consists of many gases though hydrogen prevails. Therefore we consider a nebula composed <u>only of atomic</u> hydrogen with an average density of $\bar{\rho} = 10^{-20}$ g/cm^3, an external radius of $R = 10^{17}$ cm, and an internal radius of $r_0 = 10^{10}$ cm (of course, it is equal to the radius of the core star). Here you can find physical constants which will be useful for solving this problem:

speed of light in vacuum $c \approx 3\cdot 10^8$ m/s;
elementary charge $e \approx 1{,}6\cdot 10^{-19}$ C;
electron mass $m_e \approx 9\cdot 10^{-31}$ kg;
Planck constant $h \approx 6{,}6\cdot 10^{-34}$ J·s;
permittivity of free space $\varepsilon_0 \approx 8{,}85\cdot 10^{-12}$ F/m;
Avogadro constant $N_A \approx 6\cdot 10^{23}$ mole^{-1};
Boltzmann constant $k \approx 1{,}38\cdot 10^{-23}$ J/K.
The following integrals can be useful:

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}.$$

When $b > 4$ the following approximations are valid:

$$\int_b^\infty \frac{x^3}{e^x - 1}dx \approx (b+1)^3 \cdot e^{-b} \text{ with a relative error of about } \frac{3}{(b+1)^2};$$

$$\int_b^\infty \frac{x^2}{e^x - 1}dx \approx (b+1)^2 \cdot e^{-b} \text{ with a relative error of about } \frac{1}{(b+1)^2}.$$

Part I: Star radiation and gas temperature in the nebula.

In spite of high brightness, a star can be quite accurately regarded as «blackbody», the term referring to an object absorbing almost all incident electromagnetic radiation. A hot blackbody emits electromagnetic radiation itself, its spectrum was first originally derived by M.Planck who introduced a postulate of *quantization of radiation energy*. According to this postulate the electromagnetic radiation interacts with atoms and molecules as a stream of *photons* in which the energy of a photon is related to the wave frequency ν as $E_\gamma = h\cdot\nu$. Planck proved that if a radiation with a frequency in a narrow range $(\nu,\ \nu + d\nu)$ is in equilibrium with a blackbody at a constant temperature T the energy density of the radiation equals $dW = w(\nu)\,d\nu$, where $w(\nu) = \dfrac{8\pi h}{c^3}\dfrac{\nu^3}{e^{h\nu/kT} - 1}$ is the *spectral density* of the radiation aka the Planck distribution. The total intensity of the blackbody radiation (the energy ΔE_{rad} emitted from a unit area per a unit of time) is called the total energy flux and can be determined from the Stefan-Boltzmann law: $J = \frac{\Delta E_{rad}}{\Delta S \Delta t} = \sigma \cdot T^4$. Here $\sigma \approx 5,67 \cdot 10^{-8}$ W/(m^2·K^4) is the Stefan-Boltzmann constant.

Express the Stefan-Boltzmann constant in terms of the speed of light in vacuum c, the Planck constant h, and the Boltzmann constant k.

Suppose that the whole gas cloud is in thermal equilibrium with the gas of photons emitted by the star. In this case, find the dependence of gas temperature on the distance r to the star center for $r >> r_0$. Write down the answer as a ratio $\frac{T(r)}{T_0}$, where T_0 is the temperature on the star surface.

Part II: Hydrogen atom.

Interaction of hydrogen atom and photons is described according to the Bohr model. Among the states of electron orbiting a nucleus there are *stationary states,* in such a state electron does not exchange energy with a radiation field. Absorption and emission of photons occurs during electron transition between stationary states. According to the Bohr model the energies of stationary «bound» states (electron energy levels in hydrogen atom) can be determined by employing «wave properties» of electron: the classical electron orbit corresponding to the stationary state must contain an integer number of de Broglie wavelengths $\lambda_e = \frac{h}{p_e}$, where p_e is the electron momentum. All other energies of electron are completely forbidden: an electron with a «forbidden» energy will interact with the nucleus and pass into a stationary state by exchanging energy with another objects, e.g. by emitting a photon.

Consider circular orbits of electron in hydrogen atom which contain $n = 1, 2, ...$ of de Broglie wavelengths and calculate the energy of n-th level (write down the equation as an answer; the potential energy of interaction of the electron and the nucleus tends to zero at infinity).

What is the minimal energy of a photon which is able to ionize a hydrogen atom in the lowest energy state? This quantity is also called the ionization energy (threshold). Write down the answer in electronvolts (eV). 1 eV is the unit of energy equal to the work done by electrostatic force by moving elementary charge between two points with potential difference of 1V.

It is well known that the Bohr model describes quantum systems only approximately. However, for the energy levels of hydrogen it provides (by chance) the precise answer which can be proven in the framework of the modern quantum theory. This answer is in good agreement with experimental data. The state of the lowest energy ($n = 1$) is called the ground state, and a state with $n > 1$ is called the excited state. The lifetime of atom in an excited state (i.e. the time electron spends in the stationary state with $n > 1$) does not exceed 10^{-8} s. Then the electron passes into a stationary state with a lower energy and emits a photon. The transition to a neighboring energy level ($|\Delta n| = 1$) has the highest probability, transitions with $|\Delta n| > 1$ are less probable.

Suppose the hydrogen gas is in equilibrium with the radiation of a star which surface temperature is about T_0 = 30 000 K. What is the most likely energy level (indicate its number) of hydrogen atoms at a distance $r >> r_0$ from the star? Estimate the fraction of atoms (in percent, rounded to an integer) in excited states at $r = 10^5 r_0$.

What is an approximate fraction of the star radiation energy flux consisting of photons with energies sufficient to ionize hydrogen atoms of the nebula if the temperature of the core star equals T_0 = 30 000 K? The answer should be in percent.

Part III. Radiation of nebula.

Actually, even in a stationary nebula the hydrogen is not in equilibrium with the star radiation, it «reprocesses» the radiation. The main «reprocessing» mechanism is absorption of photons with an energy exceeding the ionization threshold. As a result, the nebula has an admixture of free electrons and protons (ions H^+). Proton-electron collisions result in formation of neutral hydrogen atoms although in excited states. The rate of electron-ion collisions per unit of time is proportional to the product of their densities. Usually such a collision produces a hydrogen atom with a large n (an electron is seldom «captured» in a state with $n = 1, 2$). Electrons from these states will pass in several «steps» to the ground state by emitting photons of the hydrogen spectrum. Obviously, the corresponding frequencies will be less than the frequency of the photon absorbed by atom during ionization. Similarly, a hydrogen atom can absorb photons and pass to an excited state (for this to happen the photon energy must be very close the energy difference of the corresponding states) and then gradually return to the ground state.

Thus, the nebula absorbs a high energy photon and «converts» it into several photons of lower energies. Note that the dispersion of photon frequencies emitted during the same transition is very small: for a typical planetary nebula it is less than a hundredth of percent of the transition frequency $\nu_{n\to n'} = \dfrac{E_n - E_{n'}}{h}$. For further notice that photons emitted during transitions from a level with $n > 1$ to a level with $n' = 1$ are called «Lyman's» (the corresponding spectral lines comprise the so-called *Lyman series*) and the photons emitted from a level with $n > 2$ to a level with $n' = 2$ are «Balmer's» (the corresponding spectral lines belong to the *Balmer series*). Photons with energies below the ionization threshold and not equal to an energy difference of two stationary states are usually elastically scattered by hydrogen atoms, their frequency almost does not change.

Probability of ionization of hydrogen atom in the ground state by a photon with an energy exceeding the ionization threshold is specified by the *photoionization cross section*. This quantity has the dimension of area and is defined as a ratio of the number of photons absorbed by an atom per unit of time to the flux density of incoming photons (i.e. to the number of photons crossing a unit area of the wave front per unit of time). Let this quantity be equal to $\sigma_i \approx 10^{-17}$ cm^2 for the photons with energies near the ionization threshold. Estimate the mean free path of a photon with an energy exceeding the ionization threshold (i.e. the distance it travels in the nebula before being absorbed) assuming that the hydrogen density equals the mean density $\bar{\rho}$. Write down the answer in centimeters.

What part of the photons emitted by the star with energies exceeding the ionization threshold of hydrogen (in the ground state) is going to be «reprocessed» by the nebula into photons of lower energies? Give the answer in percent rounded to an integer.

Let ΔN be the number of photons with energies exceeding the ionization threshold of hydrogen (in the ground state) emitted by the star per a time interval Δt, ΔN_{21} be the number of photons with frequencies corresponding to the transition $(n=2)\rightarrow(n'=1)$ (the «main» line of the Lyman series) and leaving the nebula for the same time Δt, and ΔN_B be the total number of the «Balmer» photons leaving the nebula for the same time Δt. Estimate the ratio $\Delta N_{21} : \Delta N_B : \Delta N$.

An external observer studies the emission spectrum of the nebula being considered. Sketch the observed spectrum, i.e. plot the dependence $\frac{dI}{d\nu} \equiv s(\nu)$, where dI is the detected radiation intensity per a unit frequency interval $d\nu$. Indicate the main features of the dependence.

The observer found that the intensity of nebula radiation at the frequency of the main line of the Lyman series ν_{21} is 10 % of the intensity of the nebula radiation in the frequency range $\frac{2}{3}\nu_{21} \le \nu < \nu_{21}$ (the intensity of the main line in this range is not counted). What is the surface temperature T_x of the core star in the nebula? The first answer should be the equation for T_x (it must be an algebraic equation which does not contain other unknowns except T_x). The second answer should be the numerical value of T_x in Kelvin obtained by a <u>numerical solution</u> of this equation with an error not exceeding 500 K.

Now one can see that hydrogen in the nebula is partially ionized. It is interesting to find out how the degree of ionization (the ratio of the number of ionized atoms to the total number of atoms) depends on the distance to the star surface. To simplify the analysis, you may use the model based on two assumptions valid in a region of noticeable ionization:

9) a flux of photons with energies exceeding the hydrogen ionization threshold decreases mainly due to absorption of the photons by neutral atoms (rather than due to the increase of an area they spread over which);

10) hydrogen density varies (with the distance) much slower that the ionization degree, so the density can be considered approximately equal to the average density of the nebula.

Use also the information from the introduction to Part III.

Suppose that the degree of hydrogen ionization near the star surface is very high and equals 99 %. Find the distance $l_1 \equiv r_1 - r_0$ from the star surface where the degree of ionization becomes 90 % (the first answer is l_1), 50 % (the second answer is l_2), and 10 % (the third answer is l_3)? Write down all three values in cm.

Proposed solution and answers

1) The total density of radiation being in equilibrium with a surface of hot body is calculated by integrating over the whole spectrum: $W=\int_0^\infty \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}d\nu=\frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4\int_0^\infty dx\frac{x^3}{e^x-1}=\frac{8\pi^5}{15}\frac{k^4T^4}{h^3c^3}$. Here the integral given in problem statement is used. To determine the energy flux being emitted by the surface note that «equilibrium» radiation propagates at the speed of light from every point uniformly in all directions. Therefore, any element of solid angle $d\Omega$ emits the energy flux of $dJ=\frac{cW}{4\pi}d\Omega$. Let ϑ be an angle counted from a surface normal. The outgoing radiation is emitted into the angular interval $0\le\vartheta\le\frac{\pi}{2}$ and an infinitesimal solid angle between conic surfaces with apex angles of ϑ and $\vartheta+d\vartheta$ equals $d\Omega=2\pi\sin\vartheta d\vartheta$. Then the total energy flux perpendicular to the surface is $J=2\pi\frac{cW}{4\pi}\int_0^{\pi/2}\sin\vartheta\cos\vartheta d\vartheta=\frac{cW}{4}=\frac{2\pi^5k^4}{15c^2h^3}T^4$. Comparing this equation with the Stefan-Boltzmann law one obtains: $\sigma=\frac{2\pi^5k^4}{15c^2h^3}$.

2) In equilibrium nebula the energy flux emitted by the star is constant and equals to the flux from the star surface. If a gas layer at a distance $r>>r_0$ from the star were in thermal equilibrium with the radiation it would emit to «outside» the same flux as received from the «inside». According to the Stefan-Boltzmann law the flux temperature would be determined by the equation: $\sigma\cdot T^4(r)\cdot 4\pi r^2=\sigma\cdot T_0^4\cdot 4\pi r_0^2$. Hence, $\frac{T(r)}{T_0}=\sqrt{\frac{r_0}{r}}$.

3) For a circular orbit the Bohr model requires $\frac{2\pi r}{\lambda}=n\,(n\in Z)$, so $r\cdot p=m_e vr=\frac{h}{2\pi}\cdot n$. Writing down the classical equation of motion of electron in a circular orbit of a radius *r* at a velocity *v*, one obtains for hydrogen atom: $m_e\frac{v^2}{r}=\frac{e^2}{4\pi\varepsilon_0 r^2}$, then $r=\frac{e^2}{4\pi\varepsilon_0 m_e v^2}$ and $m_e vr=\frac{e^2}{4\pi\varepsilon_0 v}=n\frac{h}{2\pi}$. Therefore, the velocity and the radius of electron orbit are $v=\frac{e^2}{2\varepsilon_0 hn}$, $r=\frac{\varepsilon_0h^2n^2}{\pi m_e e^2}$ and the energy of electron in a stationary orbit is $E_n=\frac{m_e v^2}{2}-\frac{e^2}{4\pi\varepsilon_0 r}=-\frac{m_e e^4}{8\varepsilon_0^2h^2n^2}$.

4) To ionize atom, its electron must be moved far away from the nucleus, where $E=0$, i.e. to move the electron from the state with $n=1$ to a state with $n\to\infty$. Therefore, the ionization threshold of hydrogen atom is $E_i=-E_1=\frac{m_e e^4}{8\varepsilon_0^2h^2}\approx 13{,}6$ eV.

5) According to 2), the «equilibrium» gas temperature at $r = 10^5 r_0$ turns out to be $T(r) = T_0\sqrt{\dfrac{r_0}{r}} \approx 95$ K. The energy of chaotic thermal motion of hydrogen atoms at this temperature is $\dfrac{3}{2}kT \approx 0{,}012$ eV. The energy required to move electron to the lowest excited state (with n = 2) is $\Delta E_{21} = E_2 - E_1 = \dfrac{3m_e e^4}{32\varepsilon_0^2 h^2} = \dfrac{3}{4}E_i \approx 10{,}2$ eV, i.e. a thousand times more! It should be obvious that a probability of transition to an excited state is very small, practically vanishing – evidently less than one percent! The contestants familiar with the Boltzmann distribution can evaluate this probability more precisely: it is of the order of $e^{-\Delta E_{21}/kT} \approx 10^{-541}$. That is, there would be practically no excited atoms at a given distance from the star.

6) The photons able to ionize hydrogen atoms must have an energy of $h\nu \geq h\nu_i = E_i \approx 13{,}6\,\text{eV}$. For $T_0 = 30000$ K this yields $\varepsilon \equiv \dfrac{h\nu_i}{kT_0} = \dfrac{E_i}{kT_0} \approx 5{,}28$. The energy distribution of photons is given by the Planck distribution; therefore, the desired photon fraction is $\delta = \left(\int\limits_{\nu_i}^{\infty} w(\nu)d\nu\right) \Bigg/ \left(\int\limits_{0}^{\infty} w(\nu)d\nu\right)$. Using the information from the problem statement yields: $\delta = \dfrac{15}{\pi^4}\int\limits_{\varepsilon}^{\infty}\dfrac{x^3}{e^x - 1}dx \approx \dfrac{15}{\pi^4}(\varepsilon+1)^3 e^{-\varepsilon} \approx 0{,}194 \pm 0{,}015$. A reasonable answer is $\delta \sim 20$ %.

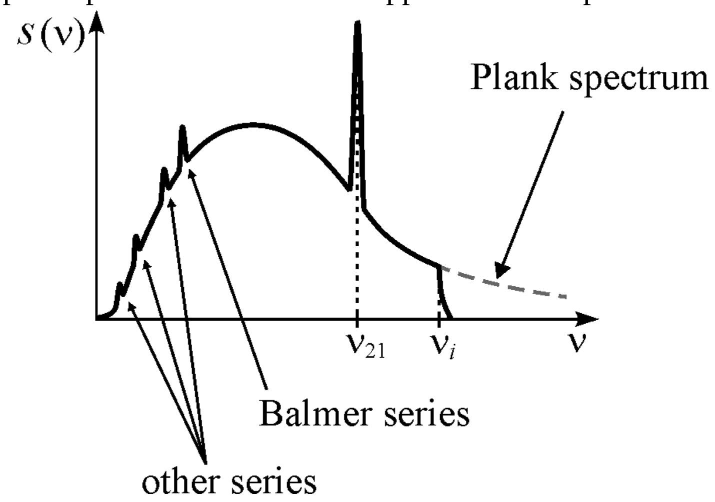
7) Density of hydrogen atoms at a mean mass density $\bar{\rho}$ is $\bar{n} = \dfrac{\bar{\rho}}{m_H}$, where m_H is the mass of a single atom. This mass can be expressed via the Avogadro constant: obviously, $m_H = \dfrac{1\text{g/mole}}{N_A} \approx 1{,}7 \cdot 10^{-24}$ g. Thus, $\bar{n} \approx 6 \cdot 10^3$ см$^{-3}$. To get an estimate one can assume that every atom creates a photon «trap» with a cross section σ_i. Then the mean free path of a photon is determined by the requirement that these «traps» overlap the entire flux cross section: $\bar{n} \cdot S \cdot l_\gamma = \dfrac{S}{\sigma_i}$, whence $l_\gamma = \dfrac{1}{\sigma_i \bar{n}} \approx 1{,}7 \cdot 10^{13}$ cm.

8) One can see that the nebula radius is about 6 thousand times greater than the «mean» free path of a photon. This means that practically all photons (all 100%, when rounded to an integer) with an energy exceeding the ionization threshold will be absorbed by the atoms and converted into photons of lower energies. *(Note that according to 6),* $\varepsilon \equiv \dfrac{E_i}{kT_0} \approx 5{,}28$, *i.e. the ionization energy is higher than the maximum* $E_m \approx 2{,}84 \cdot kT_0$ *of the photon energy distribution. So, as the energy of the photons grows, their contribution to the total flux rapidly decreases and the energy dependence of the ionization cross section could not significantly affect the result of the estimate.)*

9) According to 8), almost all photons with energies exceeding the ionization threshold are being absorbed by the nebula. Each of the absorbed photons triggers a chain of transitions. If

the first transition occurred to the state $n=1$, then a photon of Lyman series was emitted, and this photon almost certainly would be absorbed by another hydrogen atom (the majority of hydrogen atoms of the nebula are in the ground state, i.e. the energy of a Lyman photon corresponds to an allowed transition for them and the mean free path is much less than the nebula radius). The atom that had absorbed this photon would return in the same excited state. This would continue until a transition to state with $n \neq 1$ occurs. If the first transition after the capture occurred in a state with $n=2$, then a Balmer series photon would be emitted which would be inevitably followed by emission of a photon of the main Lyman series with a frequency ν_{21}. Balmer photons are practically not absorbed (since a fraction of hydrogen atoms in a state with $n=2$ is extremely low in the nebula) and leave the nebula after many elastic scattering events. Photons of the main Lyman series are absorbed and reemitted many times before leaving the nebula as well. If the first transition occured to a level with $n>2$ the situation is «repeated»: there is again a «highly excited» state of atom which would follow a transition in the state $n=1$, or in a state with $n=2$, or in a state with $n>2$. Therefore, every absorbed photon of high energy **necessarily** will «parent» a single Balmer photon and a single photon of the main Lyman series (plus several photons of other series of lesser energies). The contribution of Balmer photons and photons of the main Lyman series «originally» emitted by the star is very small due to a small width of the line and a finite density of thermal radiation. Thus, $\Delta N_{21} : \Delta N_B : \Delta N \approx 1:1:1$.

10) The spectrum of nebula radiation observed by an external observer is the spectrum of star radiation practically coinciding with the Planck spectrum corresponding to its temperature and «reprocessed» by the nebula. According to 8 and 9, the thermal spectrum will be practically «cut» from above at the frequency $\nu_i = \frac{E_i}{h}$ due to the almost complete absorption of the photons with energies above the ionization threshold. The energy emitted by this part of the spectrum is mostly transformed into the energy of photons of the main Lyman series (the energy of this photon is up to 75% of the energy of the «processed» photon), less energy goes to photons of Balmer series, and even less to low energy photons of other series. Therefore, the typical spectrum will have the approximate shape shown in the Figure.



11) In the given frequency interval $\Delta\nu$ there are **no** spectral lines of hydrogen (except for the Lyman line which contribution to the intensity is not counted), so the intensity in this frequency interval is determined by the «unprocessed» star radiation.

$$I(\Delta\nu) = \pi c r_0^2 \int_{2\nu_{21}/3}^{\nu_{21}} \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT_x} - 1} d\nu = \frac{8\pi^2 r_0^2 (kT_x)^4}{c^2 h^3} \int_z^{3z/2} \frac{x^3}{e^x - 1} dx \approx$$

$$\approx \frac{8\pi^2 r_0^2 (kT_x)^4}{c^2 h^3} \left[(z+1)^3 e^{-z} - \left(\frac{3z}{2} + 1 \right)^3 e^{-3z/2} \right].$$

Here $z \equiv \frac{2}{3} \frac{h\nu_{21}}{kT_x} \approx \frac{79000\ \text{K}}{T_x}$, and the numerical factor was written down by taking into account that the power radiated from a unit area is related to the spectral density as $J = \frac{cW}{4}$ (see 1). Note that an accurate evaluation of the factor is not necessary since it cancels out anyway. The radiation intensity at the frequency ν_{21} is determined by the condition that the number of photons with this frequency coming out of the nebula equals the number of photons with energies above the ionization threshold. Therefore,

$$I(\nu_{21}) = \pi c r_0^2 \cdot h\nu_{21} \int_{\nu_i}^{\infty} \frac{1}{h\nu} \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT_x} - 1} d\nu = \frac{8\pi^2 r_0^2 (kT_x)^3 \nu_{21}}{c^2 h^2} \int_{2z}^{\infty} \frac{x^2}{e^x - 1} dx \approx$$

$$\approx \frac{8\pi^2 r_0^2 (kT_x)^3 \nu_{21}}{c^2 h^2} (2z+1)^2 e^{-2z}.$$

This calculation takes into account that $\nu_i = \frac{4}{3}\nu_{21} = 2 \cdot \frac{2}{3}\nu_{21} \Rightarrow \frac{h\nu_i}{kT_0} = 2z$. According to the observer's data, $\frac{I(\nu_{21})}{I(\Delta\nu)} = \frac{1,5z(2z+1)^2 e^{-z/2}}{(z+1)^3 e^{z/2} - (1,5z+1)^3} = 0,1$. This is the equation for z, whence $T_x \approx \frac{79000\ \text{K}}{z}$.

This equation has to be numerically solved by evaluating the values of $f(z) \equiv \frac{1,5z(2z+1)^2 e^{-z/2}}{(z+1)^3 e^{z/2} - (1,5z+1)^3}$ for various z and comparing with 0.1 (it would me more effective to use the method of dichotomy or, if available, to use Excel or a programmable calculator and simply look at the values of $f(z)$ in the «expected» range):

z	***f(z)***
3,7	0,1623462
3,8	0,1437476
3,9	0,1275049
4,0	0,1132782
4,1	0,1007847
4,2	0,0897873
4,3	0,0800860
4,4	0,0715114

One can see, the «appropriate» value is $z \approx 4,1$. Therefore, $T_x \approx 19250$ K.

12) The equilibrium degree of ionization is achieved when the number of ionization events in a «thin layer» of gas per unit of time (equal to the number of absorbed photons with energies above the ionization threshold) equals the number of «captures» of electrons by protons

(*recombinations*). The flux of such photons decreases as the distance from the star increases not only because the surface area grows but also because of absorption: $I(r)\cdot 4\pi r^2 - I(r+dr)\cdot 4\pi(r+dr)^2 = I(r)\sigma_i n_1 \cdot 4\pi r^2 dr$. Here $n_1 = n_1(r)$ is the density of hydrogen atoms absorbing such photons (not ionized). Thus, $\frac{dI}{dr} = -\left(\frac{2}{r} + \sigma_i n_1\right) I \approx -\sigma_i n_1 I$, in accordance with the assumptions of the model used. Integration of this equation yields: $I(r) \approx I_0 \cdot \exp(-\sigma_i \int_{r_0}^{r} n_1 dr)$. The number of photons absorbed per unit time is proportional to $\sigma_i n_1 I(r)$. The number of captures is proportional to the product of electron density n_e and the density of hydrogen ions n_i which are equal because the ionized layer remains neutral. Thus, $n_1 \cdot \exp(-\sigma_i \int_{r_0}^{r} n_1 dr) = const \cdot n_e \cdot n_i = const \cdot n_i^2$. If the ionization degree is $\frac{n_i}{n} \equiv y$, then $n_1 = (1-y)n$. According to the problem statement the ionization degree varies significantly faster than the concentration of atoms, so $\frac{y^2}{1-y} \approx const \cdot \exp\left(-\frac{1}{l_\gamma}\int_{r_0}^{r}(1-y)dr\right)$, where $l_\gamma = \frac{1}{\sigma_i \bar{n}}$ is the mean free path of photons evaluated in 7. Taking the logarithm of this equation and differentiating the result, one obtains: $\left(\frac{2}{y} + \frac{1}{1-y}\right) dy = -(1-y)\frac{dr}{l_\gamma}$, i.e. $dr = -l_\gamma \left(\frac{2}{y} + \frac{1}{1-y}\right)\frac{dy}{1-y}$. Integrating the latter from the star surface to a given radius, one obtains: $r - r_0 = l_\gamma \cdot \left[2\ln\left(\frac{y_0(1-y)}{(1-y_0)y}\right) + \frac{1}{1-y_0} - \frac{1}{1-y}\right]$. Here $y_0 \approx 0{,}99$ is the given degree of ionization near the star surface. Using this formula, we find: $l_1 \approx l_\gamma[90 + 2\ln(11)] \approx 1{,}6\cdot 10^{15}$ cm, $l_2 \approx l_\gamma[98 + 2\ln(99)] \approx 1{,}8\cdot 10^{15}$ cm, and $l_3 \approx l_\gamma\left[\frac{890}{9} + 2\ln(891)\right] \approx 1{,}9\cdot 10^{15}$ cm. One can see that the ionization degree becomes less than 1% at a distance of $l = 2\cdot 10^{15}$ cm! That is, the degree of ionization remains very high at a distance of about ~1 % of the nebula radius (10^{15} cm), and drops almost to zero at a distance of ~2 %. Such a behavior confirms the assumption used in our model that the degree of ionization varies much faster than the area of the spherical surface grows and the density of the cloud decreases. Actually, it is pretty obvious that the concentration of atoms near the star is still higher than in the nebula periphery, so in the region with $r < 2\cdot 10^{15}$ m the mean free path of photons is less than the «average» value used in our calculations. Therefore, in reality the region of high ionization is even stronger «pressed» against the star.