1. As a method of finding a pair (x, y) that satisfies this equation, we can use extended Euclidian algorithm:

$$\begin{aligned} 51 &= 31\cdot 1+20, & 20 &= 51-31 \\ 31 &= 20\cdot 1+11, & 11 &= 31-20 = 31\cdot 2-51 \\ 20 &= 11\cdot 1+9, & 9 &= 20-11 = 51\cdot 2-31\cdot 3 \\ 11 &= 9\cdot 1+2, & 2 &= 11-9 = 31\cdot 5-51\cdot 3 \\ 9 &= 2\cdot 4+1, & 1 &= 9-2\cdot 4 = 51\cdot 14-31\cdot 23 \end{aligned}$$

We can now e. g. decompose $12 = 1 + 11 = 51 \cdot 13 - 31 \cdot 21$, which gives us one such pair $x_0 = 13$, $y_0 = 21$.

From the theory of linear Diophantine equations we know that all integer solutions of our equation have the form of $x = 13+31k$, $y = 21+51k$ with arbitrary integer k. Since negative k will lead to negative x and y, the minimal value of $x + y = 34 + 82k$ is 34.

2. To prove that the first three inferences are incorrect, we will provide an example of sequence $\{a_n\}$ that renders them false:

$$a_n = b_n = \frac{2+(-1)^n}{2^n},$$
$$\text{i. e.,}\quad \frac{1}{2}, \frac{3}{4}, \frac{1}{8}, \frac{3}{16}, \ldots$$

To prove that the fourth statement is correct, we will construct such sequence $\{k_m\}$, given $\{b_n\}$. In case there are infinitely many zeroes in the sequence $\{b_n\}$, we can take $\{k_m\}$ such that $b_{k_m} = 0$. Otherwise we can construct sequence $\{k_m\}$ by the following inductive process: first, we choose $k_1 = \max\{n : b_n = 0\} + 1$ (i. e., we “skip” all zeroes in $\{b_n\}$); for every next element, given k_m, we can find N such that for all $n > N$ we have $b_n < b_{k_m}$, and then take $k_{m+1} = \max(N, k_m) + 1$.

3. An example with 7 odd numbers can be constructed like this:

$$\begin{array}{ccccccc} & & & -1 & & & \\ & & -1 & & 0 & & \\ & 0 & & -1 & & 1 & \\ 1 & & -1 & & 0 & & 1 \end{array}$$

To prove that there cannot be more than 7 odd numbers, denote the numbers in the table as follows:

$$\begin{array}{ccccccc} & & & z_3 & & & \\ & & z_1 & & z_2 & & \\ & x_3 & & t & & y_3 & \\ x_1 & & x_2 & & y_1 & & y_2 \end{array}$$

Observe that among the three numbers x_1, x_2, x_3 there must be at least one even, since by $x_3 = x_1 + x_2$ they cannot all be odd. Similarly, there is at least one even number among y_1, y_2, y_3 and at least one even number among z_1, z_2, z_3. This gives us at least three even numbers.

4. Let C_i be the centers of the respective balls. Consider the right trapezium $A_1C_1C_2A_2$. By Pythagorean theorem we have

$$A_1A_2^2 + (C_1A_1 - C_2A_2)^2 = C_1C_2^2 .$$

Since $C_iA_i = r_i$ and $C_1C_2 = r_1 + r_2$, we obtain

$$A_1A_2^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2 ,$$

with similar identities for A_1A_3 and A_2A_3. This allows us to rewrite the original equations as

$$r_1r_2 = \frac{4^2}{4} = 4, \quad r_2r_3 = \frac{6^2}{4} = 9, \quad r_1r_3 = \frac{8^2}{4} = 16 .$$

Multiplying the first two equations and dividing by the third, we get $r_2^2 = 9/4$.

5. Denote the speed of the stream by v (in km/h). The boat went downstream in $30/(15+v)$ hours, and returned upstream in $30/(15-v)$ hours. We have

$$4.5 = \frac{30}{15+v} + \frac{30}{15-v} .$$

Multiplying by $(15+v)(15-v)$, we obtain

$$4.5(15^2 - v^2) = 30(15-v) + 30(15+v) ,$$

which after expansion leaves us with $4.5v^2 = 112.5$, leading to the answer $v = 5$.

6. First, let us observe that line EF is the bisector of angle AEB.

Denote by D an arbitrary point on the extension of CB beyound B (fig. 1).

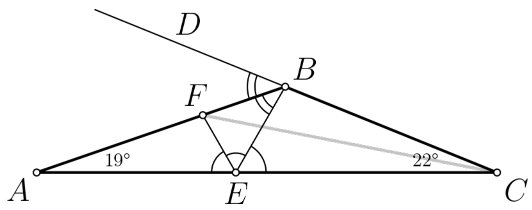

Figure 1: for the problem 6.

We have $\angle EBD = \angle ECB + \angle BEC = 82°$ and $\angle ABD = \angle ACB + \angle BAC = 41°$, which means that BA is the bisector of angle EBD.

We can now see that bisectors of external angles B and E of triangle BCE intersect at point F. Consequently, the internal bisector of the angle C of this triangle must also pass through F, which means $\angle ACF = 11°$. The answer can now be obtained by $\angle BFC = \angle ACF + \angle FAC = 30°$.

7. The angle between p and side AC is $65°$ (since it is an angle between a tangent and a chord), and the angle between q and AC is $20°$ (due to $q \parallel AB$). It follows that the angle between p and q is $65° - 20° = 45°$.

8. First, suppose that the two vertices A and B of the square that lie on the circle are not adjacent. That would imply that the line through the other two vertices is perpendicular bisector of AB, therefore contains the center of the circle, and so cannot be its tangent, contradiction. Denote the other two vertices of the square by C and D (so that the square is $ABCD$), the center of the circle by O and the point of tangency of line CD with the circle by M. It is evident that M is the midpoint of CD due to the symmetry about the perpendicular bisector of AB.

   Denote the midpoint of AB by N. We have $NA = x/2$ and $ON = NM - OM = x - 5$, where x is the side of the square. Combining this with $OA = 5$ and Pythagorean theorem we deduce $(x/2)^2 + (x-5)^2 = 5^2$, which after expansion gives us $\frac{5}{4}x^2 - 10x = 0$. Since $x = 0$ is impossible, we have $x = 8$. Thus the area of the square is $x^2 = 64$.

9. Let us denote by T the intersection of CK with AL (fig. 2), and by α the measure of angle KCA. Observe that CK contains the median to hypotenuse of the right triangle ALC, which implies $\angle TAC = \angle TCA = \alpha$. We deduce $\angle LAB = \alpha$, $\angle ABC = 90^\circ - \angle BAC = 90^\circ - 2\alpha$, and $\angle BCK = 180^\circ - 2\angle KBC = 4\alpha$. Consequently, $\angle BCA = 5\alpha$, which gives us $\alpha = 18^\circ$ and the answer $\angle ABC = 54^\circ$.

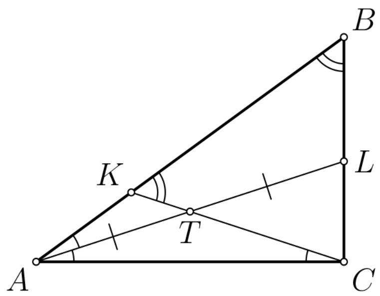

Figure 2: for the problem 9.

10. Observe that the side lengths x and y of the rectangle are the two roots of equation

$$t^2 - \frac{P}{2}t + S = 0$$

with discriminant $D = P^2/4 - 4S$, which means that, given positive P and S, such rectangle exists iff $P^2/16 \geq S$. Combining this with the relation given in the problem yields

$$\begin{gathered}\frac{P^2}{16} \geq \frac{P^2}{6} - 3P + 12\,,\\ 5P^2 - 144P + 576 \leq 0\,,\\ (5P-24)(P-24) \leq 0\,.\end{gathered}$$

Consequently, $P \in [\frac{5}{24}, 24]$.

The square of the diagonal can be found by

$$\begin{aligned}x^2 + y^2 = (x+y)^2 - 2xy &= \frac{P^2}{4} - 2S\\ &= \frac{P^2}{4} - \frac{P^2}{3} + 6P - 24 = -\frac{1}{12}P^2 + 6P - 24\,.\end{aligned}$$

This expression assumes its maximum value at parabola vertex $P = 36$, which is outside of the range $[\frac{5}{24}, 24]$. Hence, the maximum possible value is assumed at $P = 24$; we then have $S = 36$ and $x^2 + y^2 = 72$.

11. Rewriting the given system as

$$\begin{cases} (ac+bd)+(ad+bc)=13, \\ (ab+cd)+(ad+bc)=24, \\ (ab+cd)+(ac+bd)=25, \end{cases}$$

we can see that the value of $ad+bc$ can be obtained by adding the first two equations and subtracting the third. Similarly, we obtain

$$\begin{cases} ab+cd=18, \\ ac+bd=7, \\ ad+bc=6. \end{cases}$$

We can now provide the lower estimate for the sum of squares by

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+d-b-c)^2+2(ab+cd)+2(ac+bd)-2(ad+bc) \\ &= (a+d-b-c)^2+38 \geq 38\,. \end{aligned}$$

To show that this value is assumed for some a, b, c, d, we take the evidently necessary relation $a+d=b+c$ and combine it with the third of the originally given equations, giving us $a+d=b+c=\pm 5$. If we take the case of $a+d=b+c=5$ (for the sake of finding an example), we obtain $a+b+c+d=10$. This, now combined with the first two equations, gives us $\{a+b, c+d\}=\{5\pm 2\sqrt{3}\}$ and $\{a+c, b+d\}=\{4,6\}$. Taking some choices, we can deduce e. g. $a=2-\sqrt{3}$, $b=3-\sqrt{3}$, $c=2+\sqrt{3}$, $d=3+\sqrt{3}$.

12. Since $a_4+a_{16}=a_8+a_{12}=2a_{10}$, we have

$$224=a_4+a_8+a_{12}+a_{16}=4a_{10}\,,$$

which gives us $a_{10}=56$. Now, the sum in question equals to $19\cdot\frac{a_1+a_{19}}{2}=19\cdot a_{10}=1064$.

13. Note that $NK \parallel DS$. We thus have $\alpha \parallel DS$, which, combined with $\alpha \parallel AS$, gives $\alpha \parallel AD$. Since $BC \parallel AD$, the distance from B to α is the same as the distance from any other point on BC to α. Moreover, since $\alpha \parallel SAD$ and α divides SC into the ratio $2:5$, the distance from C to α is $5/7$ of the distance from C to plane SAD.

Denote the midpoints of AD and BC by U and V, respectively. Consider the section SUV of the pyramid. Since $SUV \perp SAD$ (SUV is a symmetry plane of the pyramid such that AD is orthogonally bisected by it), the distance between V and SAD equals to the distance between V and SU. We have now reduced the problem to a two-dimensional one in the plane SUV (fig. 3).

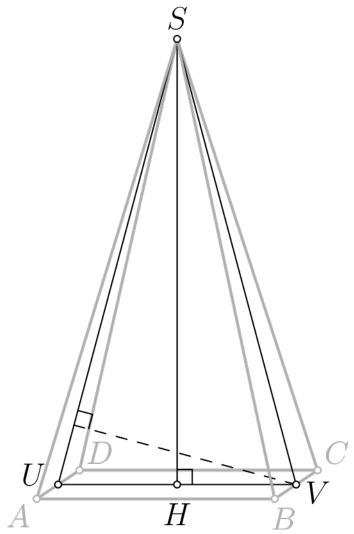

Figure 3: for the problem 13.

First, we find the side lengths of triangle SUV: $SU = \sqrt{SA^2 - AU^2} = 21\sqrt{\frac{15}{4}}$ and $UV = AB = 21$. The area of SUV is given by $\frac{1}{2} \cdot UV \cdot SH$, where H is the midpoint of UV. We have $SH = \sqrt{SU^2 - SH^2} = 21\sqrt{\frac{14}{4}}$, hence the area is $\frac{1}{2} \cdot 21 \cdot 21\sqrt{\frac{14}{4}}$. On the other hand, the same area is equal to $\frac{1}{2} \cdot SU \cdot x$, where x is the distance from V to SU. We deduce

$$x = \frac{21 \cdot 21\sqrt{\frac{14}{4}}}{21\frac{15}{4}} = 21\sqrt{\frac{14}{15}} = \frac{7\sqrt{42}}{\sqrt{5}}.$$

Since the distance in question is $\frac{5}{7}x$, the answer is $(\frac{5}{7}x)^2 = 210$.

14. Define $t = x/y$; we have $a = 1 + t$ and $b = 1 + t^{-1}$. First, let us expand $a^3 + b^3$:

$$a^3 + b^3 = (1+t)^3 + (1+t^{-1})^3$$
$$= t^3 + 3t^2 + 3t + 2 + 3t^{-1} + 3t^{-2} + t^{-3}\,.$$

Completing powers of $t + t^{-1}$, we obtain

$$(t + t^{-1})^3 + 3(t + t^{-1})^2 - 4 = 50\,.$$

By replacing $s = t + t^{-1}$ we get $s^3 + 3s^2 - 54 = 0$. Searching through possible integer roots of this equation, we can find $s = 3$; after factoring it out, we have $(s-3)(s^2 + 6s + 18) = 0$. The second factor is always positive, which means $s = 3$ is the only option.

To find the desired value, we similarly expand $a^4 + b^4$ and complete powers of s:

$$a^3 + b^3 = (1+t)^4 + (1+t^{-1})^4$$
$$= t^4 + 4t^3 + 6t^2 + 4t + 2 + 4t^{-1} + 6t^{-2} + 4t^{-3} + t^{-4}$$
$$= s^4 + 4s^3 + 2s^2 - 8s - 8$$
$$= 81 + 108 + 18 - 24 - 8 = 175\,.$$

15. Denote the hypotenuse of the triangle by c and its legs by a and b. By Vieta's formulas we have $a + b = 14 - c$ and $ab = 84/c$. This gives us $c^2 = a^2 + b^2 = (a+b)^2 - 2ab = (14-c)^2 - 2 \cdot 84/c$. Now we have an equation on c:

$$c^2 = 14^2 - 28c + c^2 - \frac{168}{c}\,,$$
$$28c - 196 + \frac{168}{c} = 0\,,$$
$$28c^2 - 196c + 168 = 0\,,$$
$$c^2 - 7c + 6 = 0\,,$$
$$(c-1)(c-6) = 0\,.$$

Since $c = 1$ is impossible (c is a largest side of the triangle, which means $14 = a+b+c \leq 3c$), we deduce $c = 6$. Now we have by Vieta $a^2 + b^2 + c^2 = (a+b+c)^2 - 2B = 14^2 - 2B$ on one hand and by Pythagoras $a^2 + b^2 + c^2 = 2c^2 = 72$ on the other, giving $B = 62$.

For completeness, we should check that the other two sides can be found from $a + b = 8$ and $ab = 14$, giving $\{a, b\} = \{4 \pm \sqrt{2}\}$.

16. When the edges were increased 2-fold, the volume got increased 8-fold, that is, by 700%.

17. We apply AM-GM inequality to prove the upper estimate:

$$x^2 y = 4 \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot y \leq 4 \left(\frac{\frac{x}{2} + \frac{x}{2} + y}{3} \right)^3 = 4 \cdot 4^3 = 256\,.$$

This value is assumed at $x = 8$ and $y = 4$.

18. Set A may contain x elements from $\{1, 3, \ldots, 37\}$ and y elements from $\{2, 4, \ldots, 36\}$ iff $1 \leq x + y \leq 4$ and x is odd; if x and y are fixed, there are $\binom{19}{x}\binom{18}{y}$ ways to choose such A. Summing over all possible pairs of x and y, we obtain

$$\binom{19}{1} + \binom{19}{1}\binom{18}{1} + \binom{19}{1}\binom{18}{2} + \binom{19}{1}\binom{18}{3} + \binom{19}{3} + \binom{19}{3}\binom{18}{1} = 37183\,.$$

19. Multiplying by $12ab$, we obtain

$$12b + 12a = ab\,,$$
$$144 = ab - 12a - 12b + 144\,,$$
$$144 = (a-12)(b-12)\,.$$

If $a \leq 12$, then $1/a + 1/b > 1/a \geq 1/12$, which is impossible. Hence $a-12$ and $b-12$ are positive integers, and $a-12 < b-12$.

Since $(a-12)^2 < (a-12)(b-12) = 144 = 12^2$, we have $a-12 < 12$. The only positive divisors of 144 that are less than 12 are 1, 2, 3, 4, 6, 8, and 9. This gives seven possible values of $a-12$; corresponding b can be found by $b-12 = 144/(a-12)$.

20. There are 10! permutations of 10 letters (if we consider all the given letters distinct). Since each word can be produced by $3! \cdot 2! \cdot 2!$ permutations that map 3 letters A, 2 letters M, and 2 letters T differently, the number of words is given by

$$\frac{10!}{3! \cdot 2! \cdot 2!} = 151200\,.$$