The 5th Olympiad of Metropolises

Mathematics

Solutions. Day 2

Problem 4. Positive numbers a, b and c satisfy $a^2 = b^2 + bc$ and $b^2 = c^2 + ac$. Prove that $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. *(Vladimir Bragin)*

Solution. We can rewrite the system in a following way

$$\begin{cases} a^2 = (b+c)b\,; \\ (b+c)(b-c) = ac\,. \end{cases}$$

Multiplying the second equation by b, we obtain

$$\begin{cases} a^2 = (b+c)b\,; \\ b(b+c)(b-c) = bac\,. \end{cases}$$

Next we replace $b(b+c)$ with a^2 in the second equation and get $a^2(b-c) = abc$.

Now we can divide both sides by a^2bc and obtain $\frac{1}{c} - \frac{1}{b} = \frac{1}{a}$. □

Another solution. By putting $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$, the condition transforms to

$$\begin{cases} \dfrac{1}{x^2} = \dfrac{1}{y^2} + \dfrac{1}{yz}\,; \\ \dfrac{1}{y^2} = \dfrac{1}{z^2} + \dfrac{1}{xz}\,; \end{cases} \quad \Rightarrow \quad \begin{cases} y^2z = x^2z + x^2y\,; \\ xz^2 = xy^2 + y^2z\,. \end{cases}$$

Transferring all terms to the right side and taking the sum of the two equations, we get

$$\begin{aligned} &0 = (x^2z + x^2y - y^2z) + (xy^2 + y^2z - xz^2) \quad \Rightarrow \\ &0 = x(xy + xz + y^2 - z^2) \quad \Rightarrow \\ &0 = x(y+z)(x+y-z)\,. \end{aligned}$$

In the last product the factors x and $y + z$ are positive, so $x + y - z = 0$, and therefore

$$\frac{1}{a} + \frac{1}{b} = x + y = z = \frac{1}{c}\,.$$ □

A one-line solution.

$$\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) \cdot abc(b+c) = a(c^2 + ac - b^2) + c(b^2 + bc - a^2) = 0\,.$$ □

Note. Observe that $b + c > a > b > c$. If one constructs a triangle ABC with sides $a = BC$, $b = CA$, $c = AB$, the equations given in the problem statement will be equivalent to $\angle A = 2\angle B$ and $\angle B = 2\angle C$, respectively. It follows that $a : b : c = \sin(\frac{4\pi}{7}) : \sin(\frac{2\pi}{7}) : \sin(\frac{\pi}{7})$, which provides another way to obtain the desired identity.

Problem 5. There is an empty table with 2^{100} rows and 100 columns. Alice and Eva take turns filling the empty cells of the first row of the table, Alice plays first. In each move, Alice chooses an empty cell and puts a cross in it; Eva in each move chooses an empty cell and puts a zero. When no empty cells remain in the first row, the players move on to the second row, and so on (in each new row Alice plays first).

The game ends when all the rows are filled. Alice wants to make as many different rows in the table as possible, while Eva wants to make as few as possible. How many different rows will be there in the table if both follow their best strategies? *(Denis Afrizonov)*

Answer: 2^{50}.

Solution. First, we prove that Eva can achieve there to be no more than 2^{50} different rows. Eva can divide every row into 50 “domino” rectangles 1×2. When Alice puts a cross in one cell of a domino, Eva puts a zero in another cell of the same domino. Upon the completion of each row every domino will be one of two types (cross-zero or zero-cross), so there will be no more than 2^{50} different rows.

Next, let us prove that Alice can achieve at least 2^{50} different rows. It is sufficient to show that as long as there are less than 2^{50} already filled rows, Alice can make the next row different from all the previous ones.

Consider the next (“new”) row that the players fill out. Let’s call *toxic* those rows among the previously filled ones that match the new row by the symbols that are already filled in it. (Thus, before Alice’s first move in the new row, all the previously filled rows will be toxic.) Let us prove that with each of her moves, Alice can reduce the number of toxic rows by half. If initially this number was less than 2^{50}, then at the end it will be less than 1, that is, the new row will not match any of the previous ones.

Suppose that Alice and Eve have each made i moves in the new row. Let us denote the current number of toxic rows by T. Each of them matches a new row by i crosses; hence, $50-i$ crosses in a toxic row are located above empty cells of the new row. Then there are exactly $(50-i)T$ crosses in toxic rows over these empty cells. There are $100-2i$ empty cells; by the pigeonhole principle, there is an empty cell over which there are at most $\frac{1}{2}T$ crosses. This is the cell in which Alice should put a cross with her $(i+1)$-th move. □

Problem 6. Consider a convex pentagon $ABCDE$. Let A_1, B_1, C_1, D_1, E_1 be the intersection points of the pairs of diagonals BD and CE, CE and DA, DA and EB, EB and AC, AC and BD, respectively. Prove that if four of the five quadrilaterals AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E, EA_1E_1A are cyclic, then the fifth one is also cyclic. *(Nairi Sedrakyan, Yuliy Tikhonov)*

Solution 1. Suppose that the quadrangles AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E are cyclic. Then $\angle ABD = \angle AB_1E = \angle EBC$, therefore, $\angle ABE = \angle CBD$. Similarly, we get that $\angle BCA = \angle DCE$ and $\angle CDB = \angle EDA$.

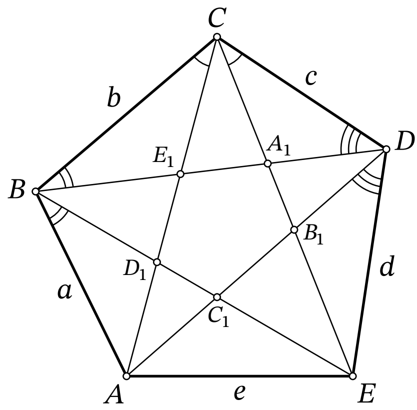

Figure 1: for the solution 1 of problem 6

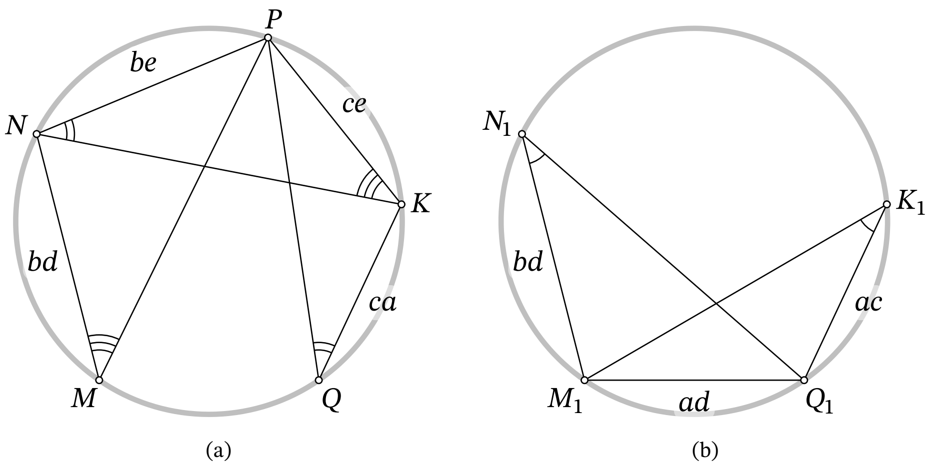

Figure 2: for the solution 1 of problem 6

Denote the lengths of the sides of the pentagon by a, b, c, d, e as shown in fig. 1. We multiply the side lengths of the triangles AED, BCD, BAE by b, e, c respectively; from the obtained triangles we can compose the shape shown in fig. 2a. We have $\angle NMP = \angle EDA = \angle CDB = \angle PKN$, therefore, the points M, N, P, K are concyclic. Similarly, we get that the points N, P, K, Q are concyclic. Hence, all five points M, N, P, K, Q are concyclic.

Now we multiply the lengths of the sides of the triangles ABC and ECD by d and a, respectively;

from the obtained triangles we can compose the figure shown in fig. 2b. It is clear that the points K_1, Q_1, M_1, N_1 are also concyclic.

It is enough to prove the equality of the arcs MN and M_1N_1, since in such case we have $\angle BAC = \angle M_1Q_1N_1 = \angle MPN = \angle DAE$, whence $\angle EAC = \angle BAD = \angle BA_1C$, that is, the quadrilateral AE_1A_1E is cyclic.

Note that $\angle BAD + \angle DEB = \angle BA_1C + \angle DE_1C = \pi - \angle ACE = \angle CAE + \angle AEC$, therefore $\angle BAC + \angle CED = \angle DAE + \angle BEA < \pi$. We have $\angle MPN + \angle QPK = \angle DAE + \angle BEA = \angle BAC + \angle CED = \angle M_1Q_1N_1 + \angle K_1M_1Q_1$. We get that $MN = M_1N_1$, $QK = Q_1K_1$ and $\widehat{MN} + \widehat{QK} = \widehat{M_1N_1} + \widehat{Q_1K_1} < 2\pi$.

Let us forget about the rest of the construction, except for the four arcs indicated in the last equality. We can move the arc QK along the circle so that its midpoint becomes diametrically opposite to the midpoint of the arc MN, and move Q_1K_1 similarly. Then $MN \parallel QK$ and $M_1N_1 \parallel Q_1K_1$, and isosceles trapezoids $MNKQ$ and $M_1N_1K_1Q_1$ are congruent, since they have the same lengths of the bases and the same angle between diagonals (the angle can be expressed as the half-sum of the arcs). Hence it follows that in the original construction $\widehat{MN} = \widehat{M_1N_1}$ as well. □

Remark. It is not hard to prove that the constructions in fig. 2 can in fact be perfectly aligned, coinciding at the corresponding points. For example, it is enough to notice that $\angle MPQ = \angle BCD - \angle DAE - \angle AEB = \angle BCD - \angle BC_1A = \angle BCA = \angle M_1N_1Q_1$.

Solution 2. As in the previous solution, suppose that the quadrilaterals AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E are cyclic and note the same three pairs of equal angles. Let us denote these angles by β, γ, δ (at the vertices B, C, D, respectively); in addition, we denote $\alpha = \angle BAC$, $\alpha_1 = \angle EAD$, $\varepsilon = \angle DEC$, $\varepsilon_1 = \angle BEA$ (fig. 3). It is enough to prove that $\alpha = \alpha_1$ and $\varepsilon = \varepsilon_1$, because the cyclicity of the quadrilateral AE_1A_1E will follow from $\angle EAC = \angle BAD = \angle BA_1C$.

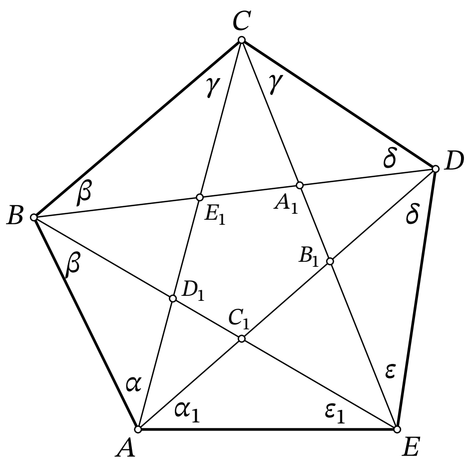

Figure 3: for the solution 2 of problem 6

For this, in turn, it is sufficient to establish that

$$\alpha + \varepsilon = \alpha_1 + \varepsilon_1 < \pi \quad \text{и} \quad \frac{\sin\alpha}{\sin\varepsilon} = \frac{\sin\alpha_1}{\sin\varepsilon_1}.$$

Indeed, considering triangles with pairs of angles α, ε and α_1, ε_1, it is easy to see that under this condition, their third angles are equal, as are the ratios of the sides adjacent to them, and the required equality of angles follows from the similarity of triangles.

As noted in the previous solution, $\alpha + \varepsilon = \alpha_1 + \varepsilon_1 < \pi$ follows from $\angle BAD + \angle DEB = \angle BA_1C + \angle DE_1C = \pi - \angle ACE = \angle CAE + \angle AEC$.

As for the ratios of the sines, we can multiply five theorems of sines:

$$1 = \frac{EA}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{CD} \cdot \frac{CD}{DE} \cdot \frac{DE}{EA} = \frac{\sin\beta}{\sin\varepsilon_1} \cdot \frac{\sin\gamma}{\sin\alpha} \cdot \frac{\sin\delta}{\sin\beta} \cdot \frac{\sin\varepsilon}{\sin\gamma} \cdot \frac{\sin\alpha_1}{\sin\delta} = \frac{\sin\alpha_1}{\sin\varepsilon_1} \cdot \frac{\sin\varepsilon}{\sin\alpha}. \quad \square$$

Solution 3. Let us introduce complex coordinates on the plane (in an arbitrary way) and identify vectors with complex numbers.

We will use the fact that the quadrilateral AE_1A_1E is cyclic if and only if

$$\frac{\overrightarrow{AE_1} \cdot \overrightarrow{A_1E}}{\overrightarrow{E_1A_1} \cdot \overrightarrow{EA}} \in \mathbb{R}.$$

Moreover, the vectors in this condition can be replaced by collinear vectors; this will not change the realness of the expression.

Denote $z_1 = \overrightarrow{AC}$, $z_2 = \overrightarrow{BD}$, $z_3 = \overrightarrow{CE}$, $z_4 = \overrightarrow{DA}$, $z_5 = \overrightarrow{EB}$. Then the above condition can be rewritten as

$$\frac{z_1 \cdot z_3}{z_2(z_1 + z_3)} \in \mathbb{R} \quad \Leftrightarrow \quad \left(\frac{1}{z_1} + \frac{1}{z_3}\right) z_2 \in \mathbb{R} \quad \Leftrightarrow$$
$$(w_1 + w_3) \cdot \overline{w_2} \in \mathbb{R},$$

where $w_i = z_i^{-1}$. For each of the rest of the five quadrilaterals its cyclicity will be equivalent to a similar condition, with a cyclic shift of indices. (For the rest of the proof we consider indices modulo 5.)

But the sum of all five expressions appearing in these conditions is always a real number:

$$\sum_{k=1}^{5} (w_k + w_{k+2}) \cdot \overline{w_{k+1}} = \sum_{k=1}^{5} (w_k \cdot \overline{w_{k+1}} + w_{k+1} \cdot \overline{w_k}) \in \mathbb{R},$$

which follows from $u \cdot \overline{v} + \overline{u} \cdot v \in \mathbb{R}$. This means that if four summands are real, then the fifth is also real. $\square$

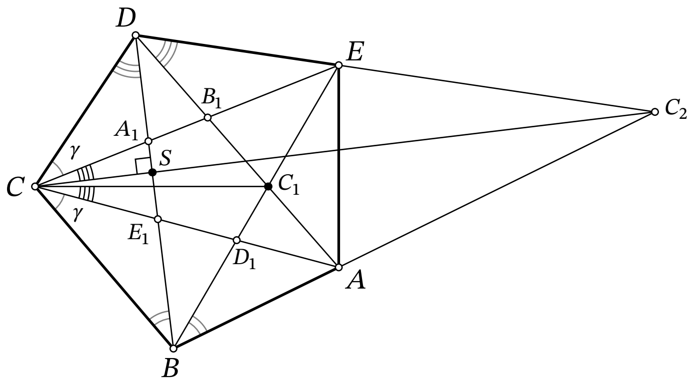

Figure 4: for the solution 4 of problem 6

Solution 4.[1] As in the first two solutions, suppose that the quadrilaterals AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E are cyclic and note the same three pairs of equal angles. In particular, denote $\gamma = \angle BCA = \angle ECD$; choose the sign of signed angles so that $\angle(EC, DC) = +\gamma$.

Denote by C_2 the intersection point of the lines BA and DE (fig. 4). Observe that

$$\angle(DE, BA) = \angle(DA, BA) + \angle(DE, DA) = \angle(EC, DB) + \angle(DB, DC) = \angle(EC, DC) = \gamma \neq 0\,,$$

which means that those lines indeed intersect. Moreover, those are the rays BA and DE that intersect (and not AB and ED), since otherwise $\angle AC_2E = \pi - \gamma > \angle ACE$ would lead to a contradiction.

By isogonal line lemma we get that CC_1 and CC_2 are isogonals with respect to angle BCD. Denote by S the intersection point of the lines CC_2 and BD. Observe that points C_1 and S lie on isogonals with respect to each of the angles ABC, BCD, CDE. It follows that C_1 and S are isogonal conjugate points with respect to the quadrilateral CBC_2D (one can prove this by considering the reflections of C_1 about the sides of the quadrilateral, and showing that they lie on a circle with center S).

But then we have $\angle CSB + \angle DSC_2 = \pi$ (a known property of points that have an isogonal conjugate with respect to a quadrilateral), which in this case means $CC_2 \perp BD$.

All that remains is to calculate some angles:

$$\angle AA_1E = \angle ABB_1 = \angle ABE + \angle EBB_1 = \angle DBC + \angle C_1CB_1 = \angle DBC + \angle E_1CC_2 = \tfrac{\pi}{2} - \gamma\,,$$
$$\angle AE_1E = \angle D_1DE = \angle ADE + \angle D_1DA = \angle CDB + \angle D_1CC_1 = \angle CDB + \angle C_2CA_1 = \tfrac{\pi}{2} - \gamma\,. \quad \square$$

[1] based on the solutions found (independently) by participants Boris Stanković and Mijail Guttierrez after the contest.