PROBLEM 1: «ABOUT ROTATION»

This problem consists of three independent parts. Notations used in any part are not related to the notations used in other parts of the problem (i.e. the same letters can be used for different physical quantities). When considering questions posed in any part of the problem, the physical quantities defined in that part should be used.

Part 1: «Launch from Rotating Planet»

There is a single planet orbiting around a star. The orbit is an ideal circle with a radius R = 200 Mkm. A «year» on this planet (the period of revolution T around the center of the star) equals 124 «days» (we define a «day» as a time interval τ between two successive moments when the star is at the zenith above some reference point on the planet equator). A «day» τ is precisely equal to 20 Earth hours, i.e. τ = 72000 s. It is known that the planet mass equals one millionth of the star mass, the planet axis of rotation is perpendicular to the orbital plane, and a planet direction of rotation around the axis is the same as a direction of its orbital rotation around the star center.

Inhabitants of the planet prepared a starship for launch. Before the launch the ship with its engines turned off was on the «planetostationary» orbit: the orbit lies at the planet equatorial plane and the ship on this orbit does not move relative to the solid surface of the planet.

Q1.1: Calculate a ratio of the planet orbit R to the radius r of the planetostationary orbit with an accuracy of better than 1%. Write down the answer as an integer.

Q1.2: Calculate the ratio of the magnitudes of maximum and minimum velocities of the ship on this orbit relative to the star center with an accuracy of better than 1%.

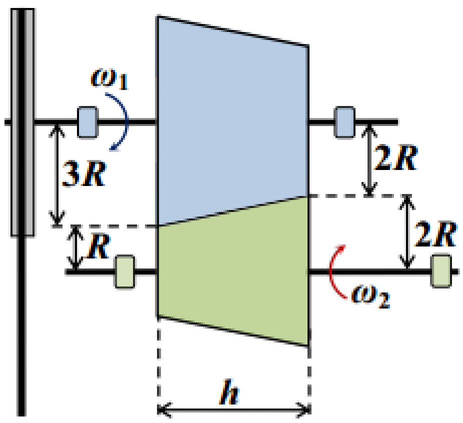
The launch has come. The engines had been turned on at the moment when the ship was going through the farthest point of «planetostationary» orbit from the star center. During the first «year» of the trip the engines were working so that the ship kept the same constant velocity it had at the launch and traveled along a straight line relative to the star.

Q1.3: Calculate the magnitude of the ship velocity in the reference frame of the planet one hour after the launch with an accuracy of better than 5%. Write down the answer in km/s.

Q1.4: Calculate the magnitude of the ship velocity in the reference frame of the planet 1 planetary year after the launch with an accuracy of better than 5%. Write down the answer in km/s and round it to an integer.

Part 2: «Friction-Type Transmission»

One of the ship mechanisms used a friction-type transmission shown in the Figure.



A chain drive rotates the «upper» axis and in a stationary regime (when angular velocities of both axes are constant) its angular velocity is ω_1 . Rotation from the «upper» to the «lower» axis is transferred by friction between two rollers shaped as truncated cones of the size shown in the Figure. The rollers are uniformly pressed against each other along the line of contact (the net force of normal reaction due to interaction between rollers is F). A coefficient of dry friction between the rollers is $\mu = 0{,}4$. The rotation axes are parallel, their directions are fixed by two pairs of virtually frictionless ball bearings. No other force except for the force exerted by the «upper» roller does not affect the rotation of the «lower» one.

Q2.1: Calculate the ratio of angular velocities $\frac{\omega_2}{\omega_1}$ in the stationary regime with an accuracy of better than 1%. Round the answer to a hundredth.

Suppose the chain drive accelerates and then maintains rotation of the «upper» axis by exerting a constant power $P = \frac{211\pi}{10}\rho R^4 h \cdot \omega_m^3$. Here ρ is a specific density of a roller material and $\omega_m \equiv 2\pi\,\text{c}^{-1} \approx 6{,}2832\ \text{s}^{-1}$ is a constant. Both rollers were at rest when the drive had turned on. Assume that all this power (except for the losses due to friction at the «lower» roller) goes into increasing kinetic energy of the «upper» roller.

Note: Equation of rotational motion, which specifies how the angular velocity of a solid object with a moment of inertia I varies under a torque M, is: $I\frac{d\omega}{dt} = M$. The moment of inertia of a uniform cone with a specific density ρ, a radius of the base R, and a height H equals $I = \frac{\pi}{10}\rho R^4 H$.

Q2.2: Derive a formula for the moment of inertia I_2 of the «lower» roller. Write down I_2 in terms of ρ and geometric size (R and h).

Q2.3: Derive a formula for the moment of inertia I_1 of the «upper» roller. Write down I_1 in terms of ρ and geometric size (R and h).

Q2.4: Determine with an accuracy of better than 15 % in which time, after the chain drive had started, the angular velocities of the rollers became equal. To do the calculations you may need the following approximation: $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, valid for $|x| << 1$. Express the answer in ms.

To answer this and the next question use the following expression for the pressure force acting between the rollers: $F = \frac{211\pi}{10}\rho R^3 h \cdot \omega_m^2$.

Q2.5: Using the results obtained in the previous questions of this part of the problem determine a value of ω_2 in the stationary regime with an accuracy of better than 3 %. Write down the answer in s^{-1}.

Part 3: «Interstellar Navigation»

In a ship's log a table of ammeter and voltmeter readings recorded in the second year of flight, when the ship executed a special maneuver, was found. During the maneuver, the ship was receding from the star along a flat spiral trajectory that intersected all radial rays pointing from the star at the same angle $\alpha = \text{arctg}(0{,}5) \approx 26{,}6°$ (see Fig.1).

I, A	2500	500	100	20	4	0,8	0,16	0,032
U, mV	785,0102	157,0034	31,3998	6,2802	1,2560	0,2512	0,0502	0,0101

The diagram in Fig.2 shows the circuit that included these ammeter and voltmeter. A photocell was always facing the star and photocell EMF was proportional to an intensity of light coming from the star. Absorption of light in interstellar space is negligible. The photocell is included in the circuit as shown in the diagram. A coil inductance $L = 50$ mH is quite small ($L << R \cdot t$, where t is a typical time, in which the ship location changes, and R is a resistance), the voltmeter and ammeter can be considered as ideal. Any information on the time of the measurements was lost, only the above readings survived.

Q3.1: Assume induction EMF to be negligible compared to photocell EMF and write down an equation (up to a constant factor) relating a current I through the coil and a distance r between the ship and the star.

Q3.2: Write down an equation for a voltage U across the coil (up to a constant factor) in terms of a distance r between the ship and the star and a radial velocity $v \equiv \frac{dr}{dt}$ of the ship.

Q3.3: Determine an angular velocity of ship rotation around the star at the beginning of the special maneuver. Write down the answer in s^{-1}.

Q3.4: Determine the angular velocity of ship rotation around the star at the end of the special maneuver. Write down the answer in s^{-1}.

Q3.5: Determine a time it took the ship to execute the special maneuver. Write down the answer in s.

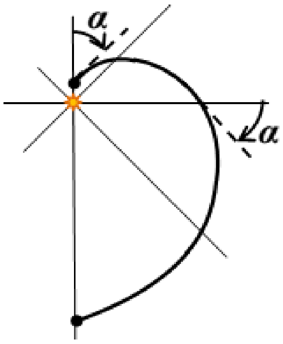


Figure 1.

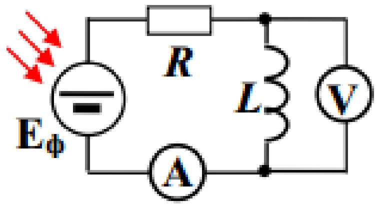


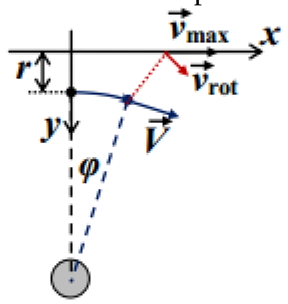
Figure 2.

PROPOSED SOLUTION:

1.1. Angular velocity Ω of planet rotation around the star is related to a linear velocity of the planet as $\Omega = \frac{2\pi}{T} = \frac{V}{R}$. Duration of a «dayplanet» depends both on an angular velocity ω of planet rotation around its axis in the «rest» reference frame (RF) of the star center and on the angular velocity of planet rotation around the star. Obviously, if a planet rotates around its axis in the same direction as it rotates around the star then under a certain relation between the angular velocities the planet will always face the star by the same side, so the day on this side will last «forever». In general, $\tau = \frac{2\pi}{\omega - \Omega} \Rightarrow \omega = \frac{2\pi}{\tau} + \frac{2\pi}{T}$. According to the problem statement, $T = 124\tau$, so $\omega = 125\Omega$. Using equation of motion for a planet of mass M orbiting a star of mass $\overline{M}$ (the equation is $M\Omega^2 R = \frac{GM\overline{M}}{R^2}$) one finds that $R^3 = \frac{G\overline{M}}{\Omega^2}$. Similarly, using the equation of motion for a ship of mass m orbiting the planet (it is $m\omega^2 r = \frac{GmM}{r^2}$) one obtains for the radius of circular orbit $r^3 = \frac{GM}{\omega^2} \Rightarrow r = \left(\frac{M}{\overline{M}}\right)^{1/3} \frac{R}{(125)^{2/3}} = \frac{R}{2500}$. One can see that $R/r = 2500$, i.e. the radius of the ship orbit is much less than the radius of the planet orbit. Obviously, the ship mass is much less than the planet mass, so the accuracy of the equation of motion written above is better than 1 %.

1.2. Linear velocity of ship rotation around the planet center is $v = \omega r = 125\Omega \cdot \frac{R}{2500} = \frac{V}{20}$. Therefore the maximum velocity of the ship at the «planetostationary» orbit is $v_{max} = v + V = \frac{21}{20}V$, and the minimum one is $v_{min} = V - v = \frac{19}{20}V$. Thus, $\frac{v_{max}}{v_{min}} = \frac{21}{19} \approx 1{,}105$.

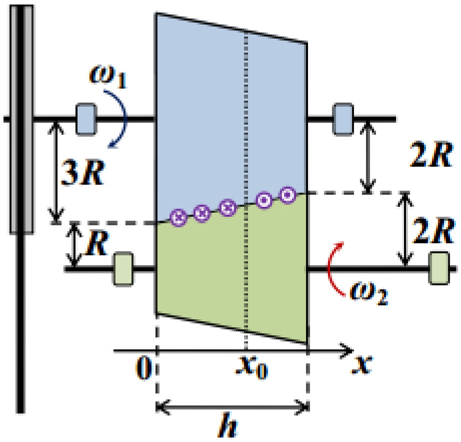
1.3. A velocity of an object (the ship) in a rotating RF equals the difference of its velocity in the rest RF and the velocity of the point which is at rest in a moving frame (the planet) and which position coincides with the object: $\vec{v}' = \vec{v}_{\max} - \vec{V} - \vec{v}_{rot}$. The rotation velocity $\vec{v}_{rot}$ is perpendicular to the radius and equals the product of the angular velocity of planet rotation around its axis and a distance from the ship to the rotation axis (see the Figure).



Let us introduce a reference frame which origin coincides with the ship position at the start. At $t = \frac{\tau}{20} = \frac{T}{2480}$ the ship traveling at a speed $v_{\max} = \frac{21}{20}V$, in the RF of the star center, has covered a distance $s = \frac{21}{20}V\frac{T}{2480} = \frac{21\pi}{24800}R$ along x. During this time the planet center has rotated around the star center by an angle $\varphi = \frac{\pi}{1240}$. Therefore it has traveled a distance $\Delta x = R\sin(\varphi) \approx \frac{\pi}{1240}R$ along x and a distance $\Delta y = R(1-\cos(\varphi)) \approx \frac{R\varphi^2}{2} \approx \frac{1}{2}\left(\frac{\pi}{1240}\right)^2 R << \Delta x$ along y. Thus, at this moment the radius connecting the planet center and the ship has an x- component equal to $r_x' = s - \Delta x = \frac{\pi}{24800}R$, and a y- component equal to $r_y' \approx r = \frac{R}{2500}$. The planet velocity has components $V_x = V\cos(\varphi) \approx V$ and $V_y = V\sin(\varphi) \approx \frac{\pi}{1240}V$, and the rotation velocity has components $v_{rot\,x} = \omega r_y' = \omega\frac{R}{2500} = \frac{V}{20}$ and $v_{rot\,y} = \omega r_x' = \omega\frac{\pi R}{24800} = \frac{5\pi}{992}V$. Note that the approximations used have an accuracy better than 1 %. Therefore the components of ship velocity relative to the planet solid surface are: $v_x' \approx \frac{21V}{20} - V - \frac{V}{20} = 0$ and $v_y' \approx -\frac{\pi}{1240}V - \frac{5\pi}{992}V = -\frac{29\pi}{4960}V$, which yields for the absolute value of this velocity $v_1 \approx \frac{29\pi}{4960}\Omega R = \frac{29\pi^2}{2480}\frac{R}{T} \approx 2{,}585$ km/s.

1.4. A «year» later the planet will return to the same position and will be moving at the same speed V along x. Meanwhile the ship will have traveled a distance $S = \frac{21}{20}V \cdot T = \frac{21\pi}{10}R$ along this axis. This distance is much greater than the component of the vector connecting the planet center and the ship ($r = R/2500$). Therefore the relative component is determined only by the y- component of angular velocity with an accuracy better than 0.1 %, i.e. $v_2 \approx \omega S = 525\pi^2\frac{R}{T} \approx 116074$ km/s. This is tremendous velocity but actually it is due to a very large distance to the planet rotation axis; in the inertial frame the ship velocity is still $\frac{21}{20}V = \frac{21\pi}{10}\frac{R}{T} \approx 148$ km/s.

2.1. Since the «lower» roller rotates in the stationary regime at a constant angular velocity, the net torque applied to the roller (due to the torques of friction forces exerted by the «upper» roller) vanishes.



The net torque of the friction forces vanishes because a surface radius varies. Introducing $x-$ coordinate of a roller cross-section one can see that a radius of «upper» roller decreases as $r_1(x) = R\left(3-\frac{x}{h}\right)$ whereas a radius of «lower» roller increases as $r_2(x) = R\left(1+\frac{x}{h}\right)$. Therefore linear velocities at different roller ends are not the same and there is sliding. Direction of the sliding is different, so the direction of friction force is different as well! Since the same force $\mu F\frac{dx}{h}$ is acting on each interval with a component of dx of the line of contact, the net torque due to these forces vanishes providing

$$\mu F\frac{R}{h}\left[\int_0^{x_0}\left(1+\frac{x}{h}\right)dx - \int_{x_0}^{h}\left(1+\frac{x}{h}\right)dx\right] = 0,$$

where x_0 is the cross-section coordinate where the rollers rotate without sliding. Evaluating the integrals one obtains an equation for $z = \frac{x_0}{h}$: $z^2 + 2z - \frac{3}{2} = 0 \Rightarrow z = \sqrt{\frac{5}{2}} - 1$. Roller radii at this cross-section are $r_1(x_0) = R\left(4-\sqrt{\frac{5}{2}}\right)$ and $r_2(x_0) = R\sqrt{\frac{5}{2}}$. Using the condition of not sliding one obtains a ratio of angular velocities: $r_1(x_0)\cdot\omega_1 = r_2(x_0)\cdot\omega_2 \Rightarrow \frac{\omega_2}{\omega_1} = 4\sqrt{\frac{2}{5}} - 1 \approx 1{,}53$.

2.2. The moment of inertia of roller 2 can be found as the difference between moments of inertia of two cones: $I_2 = \frac{\pi}{10}\rho(2R)^4 2h - \frac{\pi}{10}\rho R^4 h = \frac{31\pi}{10}\rho R^4 h$.

2.3. The moment of inertia of «upper» roller is $I_1 = \frac{\pi}{10}\rho(3R)^4 3h - \frac{\pi}{10}\rho(2R)^4 2h = \frac{211\pi}{10}\rho R^4 h$.

2.4. After the chain drive has been turned on, the upper roller starts rotating and then, due to friction, the lower one follows. It would be logical to assume that initially $\omega_1 \geq \omega_2$. If so, since the minimum radius of roller 1 equals the maximum radius of roller 2, the upper roller slides on the lower one along the whole line of contact, and a torque accelerating the lower roller equals $M_2 = \mu F\frac{R}{h}\int_0^h\left(1+\frac{x}{h}\right)dx = \frac{3}{2}\mu FR$. Equation of rotational motion of the lower roller for $\omega_1 \geq \omega_2$ is $\frac{d\omega_2}{dt} = \frac{M_2}{I_2} = \frac{15\mu F}{31\pi\rho R^3 h} = \frac{633}{155}\omega_m^2$ for given values of μ and F. Thus, $\omega_2(t) = \frac{633}{155}\omega_m^2 t$. An accelerating torque exerted on the upper roller by the chain drive is $M_{cd} = \frac{P}{\omega_1}$ and a decelerating torque due to friction is $M_1 = -\mu F\frac{R}{h}\int_0^h\left(3-\frac{x}{h}\right)dx = -\frac{5}{2}\mu FR$. Therefore the equation of rotational motion of the roller is $\frac{d\omega_1}{dt} = \frac{M_{cd} + M_1}{I_1} = \frac{10}{211\pi\rho R^4 h}\left(\frac{P}{\omega_1} - \frac{5}{2}\mu FR\right) = \omega_m^2 \frac{\omega_m - \omega_1}{\omega_1}$. After rewriting this equation as $\left(\frac{\omega_m}{\omega_m - \omega_1} - 1\right)d\omega_1 = \omega_m^2 dt$ and integration, one obtains: $\omega_m^2 t = -\omega_m \ln\left(1 - \frac{\omega_1}{\omega_m}\right) - \omega_1$. Thus, at the beginning the angular velocities of the rollers are related as $\omega_2 = -\frac{633}{155}\left[\omega_m \ln\left(1 - \frac{\omega_1}{\omega_m}\right) + \omega_1\right]$. Now one can find the common angular velocity of the rollers ($\omega_1 = \omega_2 \equiv y \cdot \omega_m$) from the equation $\frac{788}{633} y = -\ln(1-y)$ or $1 - y = e^{-\frac{788}{633}y}$. Evidently, there is a non-zero solution. Indeed, when $t > 0$, $\omega_1 > \omega_2$, and angular velocities become equal at some point. The solution can be found by trial-and-error (using the method of dichotomy) or by expanding the exponent in power series: $e^{-\frac{788}{633}y} \approx 1 - \frac{788}{633}y + \frac{1}{2}\left(\frac{788}{633}y\right)^2 - \frac{1}{6}\left(\frac{788}{633}y\right)^3 + ... = 1 - y$. Using $\tilde{y} \equiv \frac{788}{633}y$ as the variable, one obtains the equation $\tilde{y}^2 - 3\tilde{y} + \frac{465}{394} = 0$, whence $\tilde{y} = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{843}{197}} \approx 0{,}46569$. Therefore $y = \frac{633}{788}\tilde{y} \approx 0{,}3741$. Estimating the error as the largest of the discarded terms, one finds that the relative error is about $\frac{1}{12}\tilde{y}^2 \approx 2$%. (A more accurate estimate yields $y \approx 0{,}3657425 \pm 0{,}0000005$.) Using the value of y in the equation for $\omega_2(t)$, one determines the corresponding time $t = \frac{155}{633}\frac{y}{\omega_m} \approx 14{,}3$ ms.

2.5. In the stationary regime the moment $M = \frac{P}{\omega_1}$ accelerating the upper roller equals

$$\left|M_1^{fin}\right| = \mu FR\left[\int_0^z (3-w)\,dw - \int_z^1 (3-w)\,dw\right] = \frac{1}{2}\mu FR \cdot (12z - 5 - 2z^2) = 4\mu FR \cdot (\sqrt{10} - 3)$$

(a variable $w = x/h$ is used when evaluating the integral). For the given values of μ and F one obtains $\omega_1 = \frac{P}{4\mu FR \cdot (\sqrt{10} - 3)} = \frac{5\omega_m}{8(\sqrt{10} - 3)}$.

Therefore, $\omega_2 = \left(4\sqrt{\frac{2}{5}} - 1\right)\omega_1 = \frac{4\sqrt{10} - 5}{8(\sqrt{10} - 3)}\omega_m \approx 37$ s^{-1}.

3.1. For a small inductance the inductance EMF will be small and the current through the coil will be determined by the photocell EMF. Clearly, the circuit current in this case will be proportional to an intensity of incoming light and the intensity will decay according to inverse square law, i.e. $I = const \cdot \frac{1}{r^2}$. Below we write this relation as $I = I_0 \frac{r_0^2}{r^2}$.

3.2. Using this result and the equation $d\left(\frac{1}{r^2}\right) \approx -\frac{2dr}{r^3}$, one obtains:

$$U = L\left|\frac{dI}{dt}\right| = LI_0 r_0^2 \left|\frac{d(1/r^2)}{dt}\right| = 2\frac{LI_0 r_0^2}{r^3}v,$$

where $v \equiv \frac{dr}{dt}$ is a velocity of photocell recession from the star. Thus, $U = const \cdot \frac{v}{r^3}$.

3.3. Using the equation for the distance $r = r_0\sqrt{\frac{I_0}{I}}$, one finds: $U \approx 2\frac{LI^{3/2}}{r_0\sqrt{I_0}}v \Rightarrow v(t) \approx \frac{r_0\sqrt{I_0}}{2LI^{3/2}}U = \frac{1}{2L}r(t)\frac{U}{I}$. According to the table, the voltmeter and ammeter readings are related to a good degree by a formula: $U \approx I \cdot 3{,}14 \cdot 10^{-4}$ Ω. Therefore, the ratio of a ship velocity of recession from the star and a current distance between them, $\frac{v}{r} = \frac{U}{2LI} \approx 3{,}14 \cdot 10^{-3}$ s^{-1}, remains constant! On the other hand, for the motion along the specified trajectory, the ratio of the radial ship velocity v and the angular velocity ωr is equal to the cotangent of an angle between the spiral and a radius, i.e. $\frac{v}{\omega r} = \mathrm{ctg}(\alpha) \Rightarrow \omega = \frac{v}{r}\mathrm{tg}(\alpha) \approx 1{,}57 \cdot 10^{-3}$ s^{-1}. Thus, the ship orbits around the star at a constant angular speed.

3.4. At the end of the maneuver $\omega = \frac{v}{r}\mathrm{tg}(\alpha) \approx 1{,}57 \cdot 10^{-3}$ s^{-1} as well.

3.5. According to Figure 1 in the problem statement, the rotation angle varies by π during the maneuver. Then $t = \frac{\pi}{\omega} = \frac{2\pi LI}{U}\mathrm{ctg}(\alpha) \approx 2000$ s.

ANSWERS AND CRITERIA:

#	ANSWERS	Max pts
1	1.1. Answer given: 2500	**1**
	1.2. Answer given $\frac{v_{max}}{v_{min}} \approx 1{,}105$ (scoring range from **1,10** to **1,11**)	**1**
	1.3. Answer for v_1 in the range **2,35 km/s– 2,80 km/s**	**1**
	Answer for v_1 in the range **2,45 km/s– 2,70 km/s**	**+2**
	1.4. Answer for v_2 in the range **100000-130000 km/s**	**1**
	Answer for v_2 in the range **110000-122000 km/s**	**+2**
2	2.1. Answer for $\frac{\omega_2}{\omega_1}$ in the range **1,48-1,58**	**2**
	Answer for $\frac{\omega_2}{\omega_1}$ in the range **1,51-1,55**	**+2**
	2.2. The equation obtained with a numerical factor different less than a factor of three.	**0,5**
	Correct equation derived, $I_2 = \frac{31\pi}{10}\rho R^4 h$.	**+0,5**
	2.3. The equation obtained with a numerical factor different less than a factor of three.	**0,5**
	Correct equation derived, $I_1 = \frac{211\pi}{10}\rho R^4 h$.	**+0,5**
	2.4. Answer for t in the range **12 ms-17 ms**	**2**
	Answer for t in the range **14 ms-15 ms**	**+3**
	2.5. Answer for ω_2 in the range **35 s^{-1} - 39 s^{-1}**	**2**
	Answer for ω_2 in the range **36 s^{-1} – 38 s^{-1}**	**+2**
3	3.1. Equation $I = const \cdot \frac{1}{r^2}$ or its equivalent	**1**
	3.2. Equation $U = const \cdot \frac{v}{r^3}$ or its equivalent	**1**
	3.3. Answer for ω in the range **1,44·10^{-3} s^{-1} – 1,7·10^{-3} s^{-1}**	**1**
	Answer for ω in the range **1,52·10^{-3} s^{-1} – 1,62·10^{-3} s^{-1}**	**+2**
	3.4. Answer for ω in the range **1,44·10^{-3} s^{-1} – 1,7·10^{-3} s^{-1}**	**1**
	Answer for ω in the range **1,52·10^{-3} s^{-1} – 1,62·10^{-3} s^{-1}**	**+2**
	3.5. Answer for t in the range **1900 s - 2100 s**	**2**
	Answer for t in the range **1970 s - 2030 s**	**+2**
TOTAL		**35**