PROBLEM 2: «MAGNETORESISTANCE»

Magnetoresistance is the tendency of a material to change the value of its electrical resistance in an external magnetic field. Consider some mechanisms of this phenomenon.

When a constant voltage is applied across a uniform electric conductor, the distribution of charge carriers in the bulk and on the surface of the conductor changes rapidly and an ordered motion of the carriers begins. As a result, a certain stationary state emerges in which electric field distribution and carrier current density are maintained constant at any point of the conductor. A work done by electric field on acceleration of charge carriers equals, on average, an energy loss due to interaction of charge carriers with atomic lattice of the material. Since the frequency of collision with lattice defects (and therefore the effective braking force) is proportional to a carrier velocity, a stationary drift velocity $\vec{v}_{dr}$ of carrier is proportional to the accelerating force, i.e. to the electric field: $\vec{v}_{dr} = \gamma \cdot \vec{E}$. A factor γ is called the carrier *mobility* and this quantity is determined by the conductor lattice. A conductor resistivity ρ depends on the mobility and a specific density of charge carriers in the conductor. A flow of electric current in the conductor bulk is described by a current density $\vec{j}$. This vector points in the direction of a charge flow (i.e. along the velocity of positively charged carriers and in the opposite direction for negative carriers). The magnitude of $\vec{j}$ is numerically equal to a current flowing per a unit cross-sectional area. According to vector form of Ohm's law, $\vec{j} = \frac{1}{\rho}\vec{E}$.

Q1: Charge carriers of a uniform conductor have a charge e, a mobility γ , and a constant density distribution n. Write down equation for a conductor resistivity ρ in terms of these quantities.

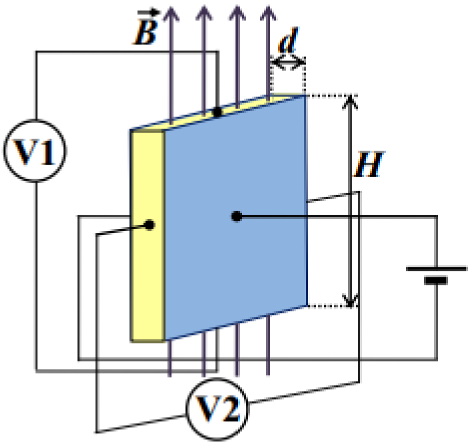
Now consider a uniform conductor with constant resistivity ρ and mobility γ. Suppose that the conductor is placed in uniform electric and magnetic fields $\vec{B} \perp \vec{E}$.

Q2: What is the angle α between the current and the field $\vec{E}$ in this conductor? The answer should be an equation for α in terms of the quantities given in the problem statement.

Q3: In this case, how does the current density j depend on the magnitude of B? Write down an equation for j in terms of the quantities given in the problem statement.

Q4: Evaluate α for $\gamma = 10$ m^2/(V·s) and $B = 0{,}1$ T. Write down the answer in degrees rounded to an integer.

Suppose a conductor has a sufficiently high resistivity $\rho = 0{,}1$ Ω·m and a mobility of its charge carriers (electrons) equal to $\gamma = 10$ м^2/(V·s). Let us also assume that a conductor material is polarized when placed in electric field and has an electric permittivity $\varepsilon = 3$. This material fills the inside volume of a flat capacitor with its plates shaped as squares of a side $H = 10$ cm separated by a small distance $d = 2$ mm. A resistivity of the plates is several orders of magnitude less than that of the conductor. A magnetic field induction $B = 0{,}1$ T is applied inside the capacitor parallel to the plates as shown in the Figure. An ideal power source applies a voltage $U = 0{,}1$ V across the plates. A couple of ideal voltmeters are connected to the conductor to the centers of opposite open sides.



Q5: Determine a voltage measured by voltmeter V1 in the stationary regime of current flow. Write down the answer in volts rounded to an integer.

Q6: Study a relation between a voltage measured by voltmeter V2 in the stationary regime of current flow and the magnetic induction B. Write down a formula for U_2 in terms of the quantities given in the problem statement.

Q7: Determine a voltage measured by voltmeter V2 in the stationary regime of current flow. Write down the answer in volts rounded to an integer.

Q8: Determine a current flowing through the power source in the stationary regime. Write down the answer in amperes rounded to an integer.

Now suppose that the conductor (with the same specifications) fills the inside volume of a cylindrical capacitor of a height H. A diameter of its internal cylinder plate is d, the diameter of the external cylinder plate is $2d$, and a resistivity of the plates is again several orders of magnitude less than ρ. A uniform magnetic field $B = 0{,}1$ T can be applied inside the capacitor parallel to the cylinder axis. A constant voltage applied across the capacitor plates equals $U = 0{,}1$ V.

Q9: Determine a current I flowing through the power source in the stationary regime in the absence of magnetic field ($B = 0$). Write down the answer in amperes rounded to an integer.

Q10: Determine a current I flowing through the power source in the stationary regime when the magnetic field is turned on. Write down the answer in amperes rounded to a hundredth.

Q11: How does an electric resistance R of the «cylindrical resistor» of our conductor depend on the magnetic induction B? The resistance is measured between the cylindrical plates. Write down an equation for R in terms of B, conductor specifications (ρ and γ), and geometric dimensions.

Q12: Determine a power P dissipated as heat in the «cylindrical resistor» when the magnetic field is turned on. Write down the answer in milliwatts rounded to a tenth.

Suppose the conductor in the cylindrical capacitor discussed above has been replaced by another one. A new conductor is not uniform. Its permittivity and electron density are constant and have the same values as those of the material discussed above. But a mobility of electrons varies in radial direction (due to modified atomic lattice) as $\gamma(r) = \gamma_0 \frac{d}{2r}$, where $\gamma_0 = 10$ m^2/(V·s) and r is a radial coordinate in the cylindrical conductor layer.

Q13: Determine a current flowing through the power source in the stationary regime when a uniform magnetic induction $B = 0{,}1$ T is applied. Write down the answer in amperes rounded to a hundredth.

Q14: When a current in a non-uniform conductor sets in a stationary regime, an electric charge is accumulated in the conductor bulk. Derive a formula for a bulk density $\varrho_{el}(r)$ of the charge in a cylindrical layer of the non-uniform conductor as a function of radius r. Write down the formula for $\varrho_{el}(r)$ as the answer. The formula must include a magnetic induction B, a current I through the conductor, conductor specifications (permittivity ε, the resistivity ρ_0 near the internal plate, and the mobility of electrons γ_0 near the internal plate), the electric constant ε_0 , and geometric dimensions.

Q15: Sketch the bulk density ϱ_{el} of electric charge accumulated inside the non-uniform conductor during transition to the stationary regime of current flow as a function of radius r. Indicate the specific points: the values at the interval endpoints, zeros (if any), and the maximum and minimum values. Take the value $\varrho_{el}(d)$ as the scale unit on the vertical axis.

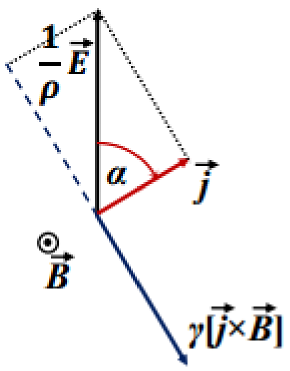
PROPOSED SOLUTION:

1. Charge carriers moving at a velocity $v = \gamma E$ through a cross-sectional area S per a time Δt occupy a volume $\Delta V = S \cdot v\Delta t$. Therefore, their net charge equals $\Delta q = en\Delta V = S\Delta t \cdot env$ and the corresponding current density is $j = \frac{\Delta q}{S\Delta t} env = en\gamma \cdot E \equiv \frac{1}{\rho} E$. Thus, $\rho = \frac{1}{en\gamma}$.

2. With magnetic field turned on, the electric carriers experience a force $\vec{F_e} = e\vec{E} + e[\vec{v} \times \vec{B}]$ and it is this force which is balanced by a «braking force» exerted by the atomic lattice. Therefore, in the presence of magnetic field Ohm's law should be written as $\vec{j} = \frac{1}{\rho}(\vec{E} + [\vec{v} \times \vec{B}])$. Recall the relation between a drift velocity of carriers and the current density, $\vec{v} = \frac{1}{en}\vec{j}$, and the formula $\frac{1}{en} = \gamma\rho$. This allows one to write down Ohm's law as an equation for current density in the given fields:

$$\vec{j} = \frac{1}{\rho}\vec{E} + \gamma[\vec{j} \times \vec{B}].$$

Clearly, all three vector terms in this equation must lie in the same plane, so vectors $\vec{j}$ and $\vec{E}$ lie in the plane orthogonal to vector $\vec{B}$:



According to this vector diagram,

$$\begin{cases} \frac{1}{\rho} E \sin\alpha = \gamma B j \\ \frac{1}{\rho} E \cos\alpha = j \end{cases} \Rightarrow \mathrm{tg}\alpha = \gamma B \Rightarrow \alpha = \mathrm{arctg}(\gamma B).$$

3. Let us square equations of this set (see **2.**) and add the results. This yields: $j = \frac{E}{\rho\sqrt{1+\gamma^2 B^2}}$.

4. For the given parameter values, $\mathrm{tg}\alpha = 1$, i.e. $\alpha = 45°$. Note that a carrier mobility unit precisely equals the inverse of a magnetic induction unit!

5. According to **4.**, vector $\vec{j}$ makes the angle $\alpha = \mathrm{arctg}(\gamma B) = 45°$ with vector $\vec{E}$ in the stationary regime for given values of the parameters. However, if the current is not perpendicular to the plates, the process is not stationary since electric charge will flow from one side of the parallelepiped to the other, perpendicular to the direction of magnetic induction. Therefore, during transition to the stationary regime a charge must build up on these sides to produce an electric field which exerts the force precisely balancing the Lorentz force: $\Delta E = vB = \gamma\rho B j$. In this way, the current will flow perpendicular to the plates and will be driven by the charges on the plates, $= \frac{1}{\rho}\frac{U}{d}$. (Note that the voltage between the plates is given and this determines the electric field. Polarizability simply results in a larger charge on the «former capacitor» plates necessary to produce the same field.) One can see that the formula relating the current density, the field, and the angle between them remains the same but the net field inside the conductor is modified since the new field component must be accounted for: $E = \sqrt{(\Delta E)^2 + \left(\frac{U}{d}\right)^2} = \frac{U}{d}\sqrt{1+\gamma^2 B^2}$. Chargers do not move along the magnetic field, so the voltage measured by voltmeter V1 is zero.

6. A voltage measured by voltmeter V2 equals $U_2 = \Delta E \cdot H = \gamma B \frac{H}{d} U$. Emergence of the transverse voltage when electric current flows in magnetic field is called the *Hall effect*. For the values given, $U_2 = \gamma B \frac{H}{d} U = 5$ V.

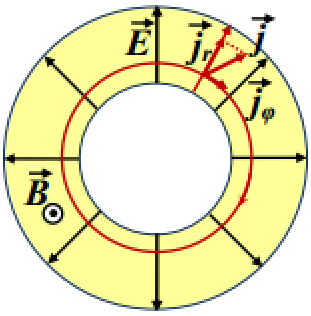
7. The current flowing through a source for the given values is $I = jH^2 = \frac{H^2 U}{\rho d} = 5$ A.

8. In the absence of magnetic field the current in a uniform conductor flows radially in any thin cylindrical layer of a radius r. If the net current equals I, the current density approximately corresponds to a uniform distribution: $j(r) = \frac{I}{2\pi r H}$. According to Ohm's law, $E(r) = \rho j(r) = \frac{\rho I}{2\pi r H}$, and a voltage between the former «capacitor plates» equals

$$U = \int_{d/2}^{d} E(r)dr = \frac{\rho I}{2\pi H}\int_{d/2}^{d}\frac{dr}{r} = \frac{\rho I}{2\pi H}\ln 2 \Rightarrow I = \frac{2\pi H}{\rho \ln 2}U.$$

Substituting numerical values, one obtains: $I = \frac{2\pi H}{\rho \ln 2}U \approx 0{,}9065$ A.

9. After magnetic field has been turned on, vector $\vec{j}$ must rotate relative to vector $\vec{E}$ by the angle $\alpha = \mathrm{arctg}(\gamma B) = 45°$. In this case the net current becomes a sum of a radial component $j_r = j \cdot \cos\alpha = \frac{j}{\sqrt{1+\gamma^2 B^2}}$ flowing between the plates and an «azimuthal» $j_\varphi = j \cdot \sin\alpha = \frac{\gamma B j}{\sqrt{1+\gamma^2 B^2}}$ flowing around a closed circle inside the conductor. Recalling that $j = \frac{E}{\rho\sqrt{1+\gamma^2 B^2}} = \frac{I}{2\pi r H}\frac{1}{\sqrt{1+\gamma^2 B^2}} = \frac{U}{\rho \ln 2}\frac{1}{\sqrt{1+\gamma^2 B^2}}\frac{1}{r}$, one finds the current flowing through the power source: $I = j_r 2\pi H r = \frac{2\pi H}{\rho \ln 2} \cdot \frac{U}{1+\gamma^2 B^2} \approx 0{,}4532$ A.



10. One can see that the net resistance of the «cylindrical resistor» in the presence of magnetic field is $R = \frac{U}{I} = \frac{\rho \ln 2}{2\pi H}(1+\gamma^2 B^2)$, i.e. resistance of a cylindrical conductor layer increases.

11. The power of heat dissipated in the conductor is $P = IU = \frac{2\pi H}{\rho \ln 2} \cdot \frac{U^2}{1+\gamma^2 B^2} \approx 45{,}3$ mW.

Note: One could say that we «forgot» the heat dissipated by the current circulating around the cylinder axis since it does not contribute to I! Actually, no mistake has been made. To see this, let us «directly» calculate the power of dissipated heat by integrating over the volume. A specific volume density of heat power is $w = \rho \vec{j}^2 = \frac{U^2}{\rho \ln^2 2} \cdot \frac{1}{1+\gamma^2 B^2}\frac{1}{r^2}$, therefore

$$P = \int_0^H dh \int_{d/2}^{d} dr\, 2\pi r\, w = \frac{2\pi H}{\rho \ln 2} \cdot \frac{U^2}{1+\gamma^2 B^2}.$$

The answer turns out to be the same.

12. Variation of electron mobility under constant electron density changes the resistivity, $\rho(r) = \frac{1}{en\gamma(r)} = \rho_0 \frac{2r}{d}$, while its value near the internal plate remains the same: $\rho_0 = \frac{1}{en\gamma_0} = 0{,}1\ \Omega\cdot$m. The radial component of the current density is related to the net current as before: $j_r(r) = \frac{I}{2\pi r H}$. This component is related to electric field at any r as

$$E(r) = \rho(r)[1 + \gamma^2(r)B^2] \cdot j_r(r) = \frac{\rho_0 I}{\pi H d}\left[1 + \gamma_0^2 B^2 \frac{d^2}{4r^2}\right],$$

And now a relation between the current and voltage is determined by the equation

$$U = \int_{d/2}^{d} E(r)dr = \frac{\rho_0 I}{\pi H d}\int_{d/2}^{d}\left[1 + \gamma_0^2 B^2 \frac{d^2}{4r^2}\right]dr = \frac{\rho_0 I}{2\pi H}\left[1 + \frac{1}{2}\gamma_0^2 B^2\right].$$

Thus, $I = \frac{4\pi H}{\rho_0}\frac{U}{2+\gamma_0^2 B^2} \approx 0{,}4189$ A.

13. A specific volume density of electric charge inside a conductor layer can be found by means of Gauss's theorem: a charge inside a cylindrical layer between radii r and $r + dr$ equals

$$dq = \varepsilon_0 \varepsilon \cdot 2\pi H[(r + dr)E(r + dr) - rE(r)],$$

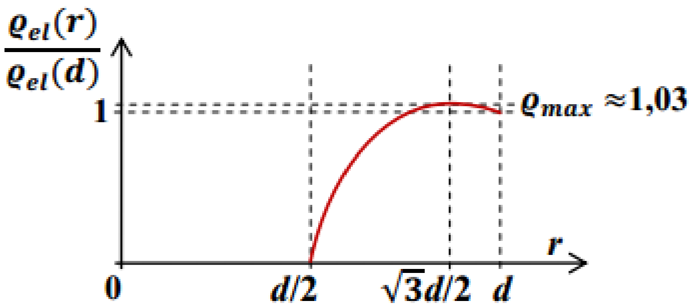
therefore

$$\varrho_{el}(r) = \frac{dq}{2\pi H r dr} = \varepsilon_0 \varepsilon \frac{1}{r}\frac{d(rE)}{dr} = \frac{\varepsilon_0 \varepsilon \rho_0 I}{\pi H d}\frac{4r^2 - \gamma_0^2 B^2 d^2}{4r^3}.$$

15. Since $\gamma_0^2 B^2 = 1$, the value near the external plate is $\varrho_{el}(d) = \frac{3\varepsilon_0 \varepsilon \rho_0 I}{4\pi H d^2}$, so the final expression is

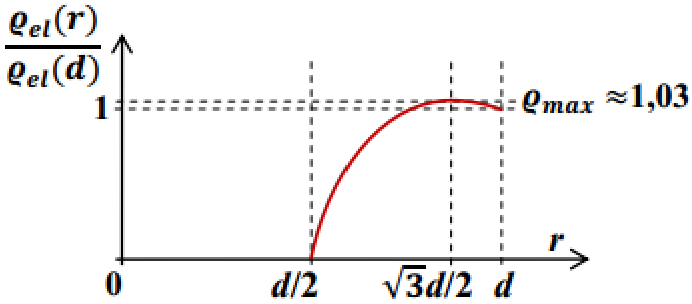
$$\varrho_{el}(r) = \varrho_{el}(d) \cdot \frac{d(4r^2 - d^2)}{3r^3}.$$

This function vanishes at $r = \frac{d}{2}$ (near the internal plate) and attains the maximum $\varrho_{max} = \varrho_{el}(d) \cdot \frac{16}{9\sqrt{3}} \approx 1{,}0264 \cdot \varrho_{el}(d)$ at $r = \frac{\sqrt{3}}{2}d$. Its approximate graph is shown below.



The influence of magnetic field on resistivity of electric conductor due to alteration of classical paths of charge carriers, which is the subject of this problem, is only a single example of this effect. Mechanisms exist by which magnetic field directly changes density and mobility of charge carriers. However, such an influence is very small for natural materials, usually it is about a variation in a hundredth or tenth of percent. In recent years artificial materials have been actively developed, e.g. those composed of thin conductor or semiconductor layers between thin ferromagnetic layers, or metamaterials possessing both the properties of semiconductors and ferromagnets. Some of these materials exhibit the effect of giant magnetoresistance (GMR): a variation of resistivity approaches 100% of a «non-magnetic» value when magnetic field exceeds a «critical» threshold. However, theoretical description of this effect relies on *quantum* properties of charge carriers and cannot be understood in the framework of classical theory. And this is completely different story…

ANSWERS AND CRITERIA:

#	ANSWERS	Max pts
1	$\rho = \frac{1}{en\gamma}$.	**1**
2	$\alpha = \mathrm{arctg}(\gamma B)$.	**2**
3	$j = \frac{E}{\rho\sqrt{1+\gamma^2 B^2}}$.	**2**
4	45°	**1**
5	0	**1**
6	$U_2 = \gamma B \frac{H}{d} U$.	**1**
7	5 B	**1**
8	5 A	**1**
9	answer in the range 0,88 A – 0,93 A	**0,5**
	answer in the range 0,90 A – 0,91 A	**+1**
10	answer in the range 0,43 A – 0,48 A	**0,5**
	answer in the range 0,45 A – 0,46 A	**+2**
11	$R = \frac{\rho \ln 2}{2\pi H}(1+\gamma^2 B^2)$.	**2**
12	answer in the range 40 mW – 50 mW	**0,5**
	answer in the range 45 mW – 45,5 mW	**+2,5**
13	answer in the range 0,40 A – 0,44 A	**0,5**
	answer in the range 0,41 A – 0,43 A	**+2,5**
14	$\varrho_{el}(r) = \frac{\varepsilon_0 \varepsilon \rho_0 I}{\pi H d} \cdot \frac{4r^2 - \gamma_0^2 B^2 d^2}{4r^3}$.	**3**
15		
	zero value at $r = d/2$	**1**
	single maximum in the range $d/2 < r < d$	**1**
	concave function in the entire region (second derivative negative everywhere)	**1**
	maximum attained at $r = \frac{\sqrt{3}}{2} d \approx 0{,}87d$	**1**
	maximum in the range 1,02 – 1,04	**1**
	TOTAL	**30**