PROBLEM 3: «QUBITS AND PHOTONS»

When working on the problem you are free to use the reference materials provided before the Olympiad.

PART 1: SINGLE-PHOTON POLARIZATION STATES

Q1: In some experiment, three photons in the same polarization state $\begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix}$ are incident on a polarizer that transmits photons polarized along x and reflects photons polarized along y. What is the probability that all three photons pass through the polarizer?

Q2: Determine the probability that two photons are transmitted and one is reflected. Photons are *identical*, i.e. they cannot be «numbered».

Q3: A coherent electromagnetic wave with intensity $I_0 = 18$ W/m^2 is incident on the same polarizer as before. All photons (there are a lot of them) of the wave are in the same state (see **Q1**). Determine an intensity of the wave (in W/m^2) transmitted through the polarizer; round the answer to an integer.

Q4: A beam of photons, all in some pure polarization state, is incident on a polarizer transmitting photons polarized along x and reflecting those polarized along y. In the first experiment, $N_0 = 1690$ photons are in the beam and $N_1 = 250$ photons passed through. Then the polarizer axis is rotated by 30° toward y. Again $N_0 = 1690$ photons in the same polarization state are incident on the polarizer (the second experiment) and this time $N_2 = 807$ pass through. The polarizer axis is rotated by another 30° in the same direction and the third experiment is done: the same number of photons in the same polarization state are sent through the polarizer. Predict the number of photons N_3 that will pass through the polarizer in the third experiment.

Note: When reconstructing a column corresponding to a pure polarization state, please, take into account that its «upper» entry can be taken to be real, i.e. the column is $\begin{pmatrix} \alpha \\ \beta_1 + i\beta_2 \end{pmatrix}$ (see the reference materials).

Q5: Estimate an accuracy of the prediction made in answering the question **Q4**. Write down you estimate of the standard deviation ΔN_3 rounded to an integer.

Q6: When a light wave passes through a certain material its polarization plane is rotated. The angle of rotation ϑ is proportional to a distance traveled by the wave in the material: $\vartheta(z) = \frac{\pi}{L} z$, where $L = 26$ cm. The light wave described in **Q3** travels through this material and then enters a polarizer. Determine the least thickness of the material for which the intensity of the wave passed through the polarizer is maximal. Write down the answer in centimeters with an accuracy of up to tenths.

Q7: Suppose a photon is in a superposition state $\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$ and some medium introduces a phase shift between polarization states of the photon. When a distance traveled by photon in the medium is z, the argument of complex number $\boldsymbol{\alpha}$ in the column $\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$ increases by $\delta = q \cdot z$ and the argument of $\boldsymbol{\beta}$ decreases by the same amount. Construct the evolution matrix for a polarization state of photon as a function of the distance z traveled by photon in this medium.

PART 2: SPIN AND HELICITY OF PHOTON

One of the most important photon «observables» related to polarization is *helicity*. Let us figure out what it means. Like many quantum particles, photon has *spin*, the «intrinsic» angular momentum. «Intrinsic» here means that this angular momentum is not related to photon motion in space, i.e. spin is non-zero in a reference frame with the origin on photon trajectory. Thus, the «intrinsic» angular momentum is due to some «internal» design of photon. It follows that a total angular momentum $\vec{J}$ of photon consists of a «regular» contribution due to photon momentum (known as «*orbital momentum*» in quantum physics) and the «intrinsic» one: $\vec{J} = [\vec{r} \times \vec{p}] + \vec{S}$. Therefore, even for a free photon the orbital momentum $[\vec{r} \times \vec{p}]$ and spin $\vec{S}$ are not separately conserved: it is the total momentum that is conserved. Obviously, momentum $\vec{p}$ of a free photon is also conserved.

A measurement of photon spin projection on a certain arbitrary direction reveals that possible values of this «observable» are $\pm\hbar$, and these correspond to two possible *spin states* of photon. It turns out, the «space» of spin states of photon and the «space» of its polarization states are actually the same two-dimensional space of photon states for a given wavevector $\vec{k}$. However, polarization is conserved for a freely propagating photon and can be used to specify its state while the projection of photon spin on an «arbitrarily chosen direction» is not conserved and cannot be used to specify photon state because it does not have a certain value. How can one choose the basic spin states of photon? The answer is to choose the spin component of photon that is conserved under free motion.

Q8: Indicate the vector, such that the component of photon spin projected on this vector is conserved under free motion of photon. In your answer, write down the notation for this vector used in the problem statement.

This conserved component of the vector of photon spin is called photon *helicity*. By convention it is measured in units of Planck's constant $\hbar$, so the allowed values of helicity are $\Lambda = \pm 1$. The states of certain helicity are represented in the polarization basis as $|+1> = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ and $|-1> = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

Q9: Construct a helicity matrix $\hat{\Lambda}$. In the answer indicate all four entries.

Q10: Determine the expectation value of photon helicity for the wave specified in **Q3**. Write down the answer with an accuracy of up to hundredths.

Q11: All photons of a photon beam with intensity I_0 propagating along z are in the same polarization state. A polarizer, which transmits photons polarized along x and reflects photons polarized along y is placed on the beam path. The intensity of the transmitted beam turns out to be $0{,}75 I_0$. Then the polarizer axis has been rotated by 45° toward y, and the intensity of the transmitted beam became equal to $0{,}5 I_0$. After that, the polarizer has been removed and replaced by a filter transmitting all photons with helicity +1 and absorbing all photons with helicity –1. The intensity of the transmitted beam became equal to $0{,}25 I_0$. Consider the polarization state in the incident beam to be mixed and determine all entries of the matrix $\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ for this state.

It should be obvious that a pure state can be considered as a special case of mixed state, when a «random» phase shift between incoherently mixed states accidentally vanishes. Therefore, sometimes a state described by matrix $\hat{\rho}$ can still be described as a column $\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$ and be actually a pure state.

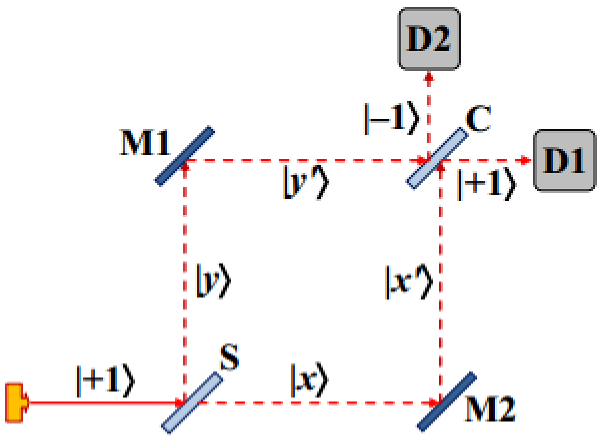
Q12: Find the condition, which entries ρ_{11}, ρ_{12}, ρ_{21}, and ρ_{22} must satisfy, so that matrix $\hat{\rho}$ describes a pure state. Write down this condition and answer the question: is the polarization state of a photon in the beam specified in **Q11** pure? Write down «YES» or «NO» in the answer.

Hint: Suppose a photon is in a pure state $\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$ and we want to find the probability to detect it in another pure state. If a matrix $\hat{\rho}$ describes the same pure state as the column $\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$ does, the desired probability can be calculated both directly (using the equation for pure states) and by means of the matrix $\hat{\rho}$.

Q13: Consider <u>another</u> photon beam propagating along z in which all photons are in the same <u>mixed</u> polarization state. We have three polarization related «observables». The first «observable» $\hat{F}$ has certain values: 1 (at the state $|x\rangle = \begin{pmatrix}1\\0\end{pmatrix}$) and 4 (at the state $|y\rangle = \begin{pmatrix}0\\1\end{pmatrix}$). The second «observable» $\hat{G}$ has certain values: +1 (at the state $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$) and –1 (at the state $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$). The third «observable» is helicity. A measurement shows that for photons of this beam the expectation value $\bar{F} = 3$, the expectation value $\bar{G} = +\frac{1}{4}$, and the expectation value of helicity is $\bar{\Lambda} = -\frac{1}{4}$. Determine all entries of matrix $\hat{\rho} = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}$ for this state.

PART 3: QUANTUM ALGORITHM OF BOMB TESTING

Consider a setup shown in the Figure.



A source emits single photons in the state of certain helicity equal to +1 and directs them to a semi-transparent beam-splitting mirror. The mirror transmits photons polarized along x (then they reach an ideal mirror M2) and reflects photons polarized along y (then they reach an ideal mirror M1). Ideal mirrors change direction of propagation of a photon and change its phase by π, the latter can be described as multiplication of photon column by –1. Photons reflected by ideal mirrors hit a semi-transparent mirror-cooperator C. The cooperator transforms the polarization state of incident photon into $|f\rangle = |1\rangle + |2\rangle$, where $|1\rangle$ and $|2\rangle$ are polarization states corresponding to waves of probability of photons to come from M1 and M2, respectively. A photon passed through the cooperator goes to a detector D1 if its helicity equals +1 and to a detector D2 if its helicity equals –1.

It is important that the separator splits the probability wave of an incident photon: after the separator the photon is in a state with some probability to go through a channel with M1 and with an «additional» probability to go through a channel with M2. The ideal mirrors are regarded as «firmly» fixed: they do not exchange energy with a photon and do not destroy coherence of the probability waves, so before D1 or D2 has fired we do not know what happened to photon inside the installation, the photon remains in a pure state all the time.

Q14: Determine the probabilities for detectors D1 and D2 to fire after a photon has been emitted by the source. In your answer write down the probabilities in turn (first w_1 and then w_2) with an accuracy of up to hundredths.

Now a technological problem. Suppose we have an «unlimited» supply of bombs equipped by a «single-photon detonator». This detonator is a mirror reflecting all incident photons and registering the recoil momentum. When the mirror reflects even a single photon, the detonator makes the bomb go off. After the detonator has been installed it cannot be removed without damaging it. It is known though that a half of the bombs (nobody knows, which ones) have a malfunctioning detonator: the mirror reflects a photon but does not «feel» the recoil and the bomb does not explode regardless of the number of photons reflected.

There is a single way to check whether a detonator properly operates: to figure out whether it detects a reflected photon. We have to find a bomb with certainly operating detonator without exploding it (and we do not care how many bombs and another equipment we will explode in the process).

Obviously, the problem has no solution from the classical physics perspective: if we direct a photon to a functioning detonator, the bomb will go off and after a large series of the tests we will have only the bombs with malfunctioning detonators. Let use for testing the installation from **Q14** in which the mirror M2 is replaced with the mirror-detonator of a bomb. A malfunctioning detonator can be considered as the ideal mirror M2. A properly operating detonator actually does a measurement by taking some photon momentum. A properly operating detonator destroys coherence of probability waves going in different channels of the installation and after the interaction, it turns out, the photon goes along the channel where the interaction happened with the probability 1 while the polarization state of the photon remains the same as it was before the interaction with the detonator.

Q15: Determine the probability that after photon emission in the setup with properly functioning detonator M2 the bomb does not explode and D1 fires. Determine also the probability that the bomb does not explode and D2 fires. In your answer, write down the probabilities in turn (first w_1 and then w_2) with an accuracy of up to hundredths.

PROPOSED SOLUTION:

1. The transmission probability of a single photon $w_x = |\boldsymbol{\alpha}|^2 = \left|\frac{1}{3} + i\frac{2}{3}\right| = \frac{5}{9}$. All three will pass with a probability $\left(\frac{5}{9}\right)^3 = \frac{125}{729} \approx 0{,}1715$.

2. Two photons pass with a probability $\frac{5}{9}$, a single photon is reflected with a probability $\frac{4}{9}$, and there are three ways to select a reflected photon, thus the probability of the specified event is $3 \cdot \left(\frac{5}{9}\right)^2 \frac{4}{9} = \frac{100}{243} \approx 0{,}4115$.

3. Obviously, $I = w_x I_0 = \frac{5}{9} I_0 = 10$ W/m^2.

4. According to the hint, we take α to be real. Using the normalization condition, $\alpha^2 + |\boldsymbol{\beta}|^2 = 1$, we obtain: $\beta = \sqrt{1-\alpha^2} \cdot e^{i\varphi}$, where φ is a phase to be determined. From the result of the first experiment an estimate follows: $w_x = |\alpha|^2 = \frac{N_1}{N_0} = \frac{25}{169}$ and hence $\alpha = \frac{5}{13}$. Thus, $\beta = \frac{12}{13} \cdot e^{i\varphi}$. In the second experiment we measure the probability that photon is in the polarization state $|x'\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$. It is equal to

$$\frac{N_2}{N_0} = \frac{807}{1690} = \left|\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 5/13 \\ 12e^{i\varphi}/13 \end{pmatrix}\right|^2 = \frac{219 + 120\sqrt{3}\cos(\varphi)}{676}.$$

From this equation we obtain $\cos(\varphi) = \frac{173}{200\sqrt{3}} \approx 0{,}499$. The process is random and the statistical fluctuations are surely greater than 0,1 %, so this quantity is practically indistinguishable from 0,5 within the error margin and $\varphi \approx \pi/3$. Thus the photon polarization state is described by a column $\begin{pmatrix} 5/13 \\ \frac{12}{13} \cdot e^{i\pi/3} \end{pmatrix}$ to a good accuracy. In the third experiment, $|x''\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$ and the photon transmission probability is

$$w'' \approx \left|\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{13} \\ \frac{12}{13} \cdot e^{\frac{i\pi}{3}} \end{pmatrix}\right|^2 = \frac{457 + 60\sqrt{3}}{676} \approx 0{,}8298.$$

Therefore the expectation value is $N_3 = w'' \cdot N_0 \approx 1402$.

5. According to a formula in the reference materials, $\Delta N_1 = \sqrt{1690 \cdot \frac{25}{169}\frac{144}{169}} \approx 15$ for the first experiment. Hence, the relative error of w_x is about $\frac{\Delta N_1}{N_1} \approx 6$ %. Similarly, $\Delta N_2 = \sqrt{1690 \cdot \frac{807}{1690}\frac{883}{1690}} \approx 20{,}5$ and here the error is approximately $\frac{\Delta N_2}{N_2} \approx 2{,}5$ %. These figures are used to determine the transmission probability in the third experiment, its relative error is $\sqrt{(0{,}06)^2 + (0{,}025)^2} = 0{,}065 = \frac{\Delta N_3}{N_3}$, i.e. about 6,5 %. Thus, $\Delta N_3 \approx 91$ and $N_3 \approx 1400 \pm 90$.

6. After a photon passes the layer of the medium, its polarization state (see the reference materials) becomes:

$$\begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \begin{pmatrix} (1+2i)/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} (\cos(\vartheta) + 2\sin(\vartheta) + 2i\cos(\vartheta))/3 \\ (\sin(\vartheta) - 2\cos(\vartheta) + 2i\sin(\vartheta))/3 \end{pmatrix}.$$

Therefore, the probability to pass through the polarizer equals

$$w_x = \frac{1}{9}\{[\cos(\vartheta) + 2\sin(\vartheta)]^2 + 4\cos^2(\vartheta)\} = \frac{1}{2} + \frac{1}{18}[\cos(2\vartheta) + 4\sin(2\vartheta)] = \frac{1}{2} + \frac{\sqrt{17}}{18}\cos(2\vartheta - \gamma),$$

where $\gamma \equiv \mathrm{arctg}(4)$. It is not difficult to see that the maximum $w_{x\,max} = \frac{9+\sqrt{17}}{18} \approx 0{,}729$ is at the least rotation angle $\vartheta_m = \frac{\gamma}{2} = \frac{1}{2}\mathrm{arctg}(4) \approx 38°$. Let us evaluate the required minimum thickness of the layer: $\frac{1}{2}\mathrm{arctg}(4) = \frac{\pi}{L} z_m \Rightarrow z_m = \frac{\mathrm{arctg}(4)}{2\pi} L \approx 5{,}5$ cm.

7. We are looking for an evolution matrix $\hat{U}(z)$, such that $\hat{U}(z)\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\alpha e^{+i\delta}}\\ \boldsymbol{\beta e^{-i\delta}}\end{pmatrix}$ where $\delta = qz$. One can see that

$$\hat{U}(z) = \begin{pmatrix} e^{iqz} & 0 \\ 0 & e^{-iqz} \end{pmatrix}.$$

8. Both the total angular momentum and the momentum for a single photon are conserved, therefore the projection of the total angular momentum on the momentum is conserved as well, i.e. $(\vec{J}\cdot\vec{p}) = ([\vec{r}\times\vec{p}]\cdot\vec{p}) + (\vec{S}\cdot\vec{p}) = const$. Obviously, the first term is zero (as a dot-product of orthogonal vectors), so the projection of the photon spin on the direction of its momentum is conserved. Since $\vec{p} = \hbar\vec{k}$, the spin projection on the direction of wavevector is conserved as well. Thus, a correct answer is $\vec{p}$ or $\vec{k}$.

9. Using the spectral decomposition formula for the helicity matrix, one obtains:

$$\hat{\Lambda} = (+1)\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix} + (-1)\cdot\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

10. A photon in this wave has a helicity +1 with a probability

$$w_+ = \left|\begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}(1+2i)/3\\ -2/3\end{pmatrix}\right|^2 = \frac{1}{18}|1+4i|^2 = \frac{17}{18}.$$

Obviously, the negative helicity comes with a probability $w_- = 1 - w_+ = \frac{1}{18}$. Therefore,

$$\bar{\Lambda} = (+1)\frac{17}{18} + (-1)\frac{1}{18} = \frac{8}{9} \approx 0{,}89.$$

11. The diagonal entries of matrix $\hat{\rho}$ are $\rho_{11} = w_x = 0{,}75$ and $\rho_{22} = w_y = 0{,}25$. Let us introduce a notation $\rho_{12} \equiv b\cdot e^{i\varphi}$. Then $\rho_{21} = (\rho_{12})^* = b\cdot e^{-i\varphi}$. A polarization state of a photon that passed through the polarizer after its axis has been rotated is $|x'\rangle = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$. Hence, the photon transmission probability after the rotation is

$$0{,}5 = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}0{,}75 & b\cdot e^{i\varphi}\\ b\cdot e^{-i\varphi} & 0{,}25\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = 0{,}5 + b\cdot\cos(\varphi).$$

Therefore, $b\cdot\cos(\varphi) = 0$. The same photon has a helicity +1 with a probability

$$0{,}25 = \begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}0{,}75 & b\cdot e^{i\varphi}\\ b\cdot e^{-i\varphi} & 0{,}25\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = 0{,}5 - b\cdot\sin(\varphi),$$

and $b\cdot\sin(\varphi) = 0{,}25$. It follows from these equations that $b = 0{,}25$ and $\varphi = \frac{\pi}{2}$. Using the identity $e^{i\varphi} = i$, one obtains :

$$\hat{\rho} = \begin{pmatrix}0{,}75 & 0{,}25i\\ -0{,}25i & 0{,}25\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & i\\ -i & 1\end{pmatrix}.$$

12. Suppose a mixed state defined by a matrix $\hat{\rho} = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}$ is actually pure, i.e. can be also described by a column $\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$. Consider another pure state described by a column $\begin{pmatrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{pmatrix}$. Then the probability that a photon in the mixed state $\hat{\rho}$ will be detected in the pure state $\begin{pmatrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{pmatrix}$ equals

$$(\boldsymbol{\alpha}'^* \quad \boldsymbol{\beta}'^*)\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{pmatrix} = \rho_{11}\boldsymbol{\alpha}'^*\boldsymbol{\alpha}' + \rho_{12}\boldsymbol{\alpha}'^*\boldsymbol{\beta}' + \rho_{21}\boldsymbol{\beta}'^*\boldsymbol{\alpha}' + \rho_{22}\boldsymbol{\beta}'^*\boldsymbol{\beta}'.$$

If the column $\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$ corresponds to the same state as $\hat{\rho}$, the above probability can be written as

$$\left|\begin{pmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{pmatrix}^\dagger \begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}\right|^2 = (\boldsymbol{\alpha}'^*\boldsymbol{\alpha} + \boldsymbol{\beta}'^*\boldsymbol{\beta})(\boldsymbol{\alpha}'\boldsymbol{\alpha}^* + \boldsymbol{\beta}'\boldsymbol{\beta}^*) =$$
$$= \boldsymbol{\alpha}^*\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'^*\boldsymbol{\alpha}' + \boldsymbol{\beta}^*\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'^*\boldsymbol{\beta}' + \boldsymbol{\alpha}^*\boldsymbol{\beta}\cdot\boldsymbol{\beta}'^*\boldsymbol{\alpha}' + \boldsymbol{\beta}^*\boldsymbol{\beta}\cdot\boldsymbol{\beta}'^*\boldsymbol{\beta}'.$$

The expressions on the right in the last two equations coincide providing $\hat{\rho} = \begin{pmatrix}\boldsymbol{\alpha}^*\boldsymbol{\alpha} & \boldsymbol{\beta}^*\boldsymbol{\alpha}\\ \boldsymbol{\alpha}^*\boldsymbol{\beta} & \boldsymbol{\beta}^*\boldsymbol{\beta}\end{pmatrix}$. One can see that the necessary consequence of this requirement is a relation $\rho_{11}\rho_{22} - \rho_{12}\rho_{21} = 0$ (those who are familiar with theconcept of determinant see that this is the requirement for the determinant of $\hat{\rho}$ to vanish). The matrix $\hat{\rho} = \frac{1}{4}\begin{pmatrix}3 & i\\ -i & 1\end{pmatrix}$ does not satisfy to this requirement, so the photon state at question is not pure: the answer is «NO».

13. States with certain values of F are the basic states $|x\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|y\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, so $\bar{F} = 3 = \rho_{11}\cdot 1 + (1-\rho_{11})\cdot 4 = 4 - 3\rho_{11} \Rightarrow \rho_{11} = \frac{1}{3}$ and $\rho_{22} = 1 - \rho_{11} = \frac{2}{3}$.

Let us write $\rho_{12} \equiv b\cdot e^{i\varphi}$ and $\rho_{21} = b\cdot e^{-i\varphi}$ as in **Q11**. Let $\tilde{w}_+$ be the probability that $G = +1$. Then $\bar{G} = +\frac{1}{4} = (+1)\cdot\tilde{w}_+ + (-1)\cdot(1-\tilde{w}_+) \Rightarrow \tilde{w}_+ = \frac{5}{8}$. On the other hand,

$$\tilde{w}_+ = \frac{5}{8} = \left(1/\sqrt{2} \quad 1/\sqrt{2}\right)\cdot\begin{pmatrix}1/3 & b\cdot e^{i\varphi}\\ b\cdot e^{-i\varphi} & 2/3\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{1}{2} + b\cdot\cos(\varphi).$$

Thus, $b\cdot\cos(\varphi) = \frac{1}{8}$.

Let us repeat the above steps for helicity:

$$\bar{\Lambda} = -\frac{1}{4} = (+1)w_+ + (-1)(1-w_+) \Rightarrow w_+ = \frac{3}{8},$$
$$w_+ = \frac{3}{8} = \left(1/\sqrt{2} \quad -i/\sqrt{2}\right)\cdot\begin{pmatrix}1/3 & b\cdot e^{i\varphi}\\ b\cdot e^{-i\varphi} & 2/3\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = \frac{1}{2} - b\cdot\sin(\varphi) \Rightarrow b\cdot\sin(\varphi) = \frac{1}{8}.$$

Therefore, $b = \frac{1}{4\sqrt{2}}$ and $\varphi = \frac{\pi}{4}$. Using the identity $e^{i\varphi} = \frac{1+i}{\sqrt{2}}$, we finally obtain

$$\hat{\rho} = \begin{pmatrix}1/3 & (1+i)/8\\ (1-i)/8 & 2/3\end{pmatrix}.$$

14. Since $|+1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$, the photon after having passed through the separator goes in the channel M1 or M1 with a probability 0,5. It follows that polarization of probability waves in these channels is described by the columns $\begin{pmatrix}0\\ \frac{i}{\sqrt{2}}\end{pmatrix}$ and $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$, respectively. Reflection by ideal mirror multiplies the columns by –1, so the waves with polarization states $|1\rangle = \begin{pmatrix}0\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$ and $|2\rangle = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$ will come to the cooperator. Hence, after the cooperator the photon polarization state is $|f\rangle = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix} = -|+1\rangle$! This vector satisfies $\hat{\Lambda}|f\rangle = (+1)\cdot|f\rangle$, i.e. it has the helicity +1 and therefore necessarily reaches detector D1. Thus, $w_1 = 1$ and $w_2 = 0$. No photon reaches D2.

15. If a detonator is malfunctioning, the sequence of events is the same as in the above scenario. We send several photons, see that a bomb does not go off and all photons reach D1, and conclude that with a high probability the bomb has a malfunctioning detonator. If the detonator is operating properly, the bomb explodes with a probability 0,5 (when the photon goes in the channel with the detonator) and we have to start anew. The photon goes in the channel with M1 with a probability 0,5. However, as long as the photon remains in a pure state, the probability wave goes through both channels and although the photon is not present in the detonator channel, the probability wave interacts with the detonator. The detonator has not fired and we now KNOW that the photon went through the channel with M1 and nothing comes from the detonator channel to the cooperator, i.e. now $|1\rangle = \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$! Therefore, in this case

$$|f\rangle = \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \left(-\frac{1}{2}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} + \left(-\frac{1}{2}\right)\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \left(-\frac{1}{2}\right)|+1\rangle + \left(-\frac{1}{2}\right)|-1\rangle.$$

One can see that in a not exploded installation with operating detonator the photon goes to D1 with a probability 0,25 and with the equal probability to D2. This solves the technological problem: if, after a photon has been sent, the bomb does not explode and detector D2 fires, this proves that the detonator operates properly! We see that quantum object, a photon, has «sensed» the presence of operating detonator although the photon was not there (but it could had been). The answer to this question is $w_1 = 0{,}25$ and $w_2 = 0{,}25$.

ANSWERS AND CRITERIA:

#	ANSWERS	Max pts
1	0,1715 (scoring range from 0,170 to 0,173).	**1**
2	0,4115 (scoring range from 0,40 to 0,42)	**2**
3	10 W/m^2	**1**
4	Answer in the range from 1370 to 1430	**2**
	Answer in the range from 1390 to 1410	**+2**
5	ΔN_3 in the range from 70 to 110	**1**
	ΔN_3 in the range from 80 to 100	**+1**
6	Answer in the range from 5 cm to 6 cm	**1**
	Answer in the range from 5,4 cm to 5,6 cm	**+1**
7	$\widehat{U}(z) = \begin{pmatrix} e^{iqz} & 0 \\ 0 & e^{-iqz} \end{pmatrix}$	
	Zero non-diagonal entries	**0,5**
	Correct diagonal entries	**+0,5**
8	$\vec{p}$ or $\vec{k}$	**1**
9	$\widehat{\Lambda} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	**3**
10	0,89 or $\frac{8}{9}$	**2**
11	$\hat{\rho} = \begin{pmatrix} 0{,}75 & 0{,}25i \\ -0{,}25i & 0{,}25 \end{pmatrix}$	**2**
12	equation $\rho_{11}\rho_{22} - \rho_{12}\rho_{21} = 0$ (or equivalent) derived	**1**
	NO	**+1**
13	$\hat{\rho} = \begin{pmatrix} 1/3 & (1+i)/8 \\ (1-i)/8 & 2/3 \end{pmatrix}$	**3**
14	$w_1 = 1, \quad w_2 = 0$	**2×1,5=3**
15	$w_1 = 0{,}25, \quad w_2 = 0{,}25$	**2×3=6**
	TOTAL	**35**