The 6th Olympiad of Metropolises

Mathematics

Marking schemes

Day 1

Problem 1. *Marking scheme*

Maximum of the following points is taken:

7 p. for a complete solution;

4 p. for a solution with binary notation (like the third solution) that ignores the problem with the second bit.

1 p. A chain $a \cdot b \to (a+1) \cdot b \to (a+1) \cdot (b+1) \to (a+2) \cdot (b+1) \to (a+2) \cdot (b+2) \to \ldots$ is considered.

Points are deducted for the following flaws in an otherwise correct solution:

−0 p. if the solution does not work for some small n (a finite set);

−0 p. every two consecutively added divisors are in fact different, but the proof of this fact is incomplete;

−1 p. if some consecutively (like in the second solution) added divisors are equal, but it can be easily remedied by swapping some consecutive steps.

−1 p. if upon adding a divisor that is the largest power of two (like in the fourth solution) it is stated that the largest power of two by which the number is divisible increases *exactly* by 1.

Problem 2. *Marking scheme*

Maximum of the following points is taken:

7 p. for a complete solution;

1 p. for conjecturing that the lines B_1P_1 and C_1Q_1 intersect at the base of the altitude from A;

The following advancements are not awarded with points:

0 p. ratios of some segments are equated (for example, using the property of the bisector, similarity of triangles ABB_1, ACC_1, APP_1, AQQ_1, similarities of triangles PP_1Q and CC_1Q, etc.), but the paper does not include Menelaus theorem nor any other way to equate the ratios in which lines B_1P_1 and C_1Q_1 divide the segment PQ;

0 p. for incomplete computations (coordinates, complex numbers, vectors, trigonometry, etc.).

Problem 3. *Marking scheme*

Maximum of the following points is taken:

- 7 p. for a complete solution;
- 1 p. for obtaining an equality expressing b_{i-1} through b_i, a_{i-1} and product $A = a_1 a_2 \dots a_n$, like in the second solution; or an equivalent relation between b_{i-1}, b_i, a_{i-1} and A.
- 2 p. for summing up the above mentioned equality for all i in a way similar to that of the second solution and cancelling out terms A and $2Aa_{i-1}$.

The following advancements are not awarded with points:

- 0 p. for considering particular cases of small n.
- 0 p. for just summing up all equalities, without first reducing them to the shortened form.
- 0 p. for considering trivial cases like when $a_i \geq 1$ for some i.

Points are deducted for the following flaws in an otherwise correct solution:

- −1 p. if the solution does not work in the case when $a_i \geq 1$ for some i.

Day 2

Problem 4. *Marking scheme*

Consider the bounds $A < s_{11} < B < s_{15} < C$, where A are all sums with x_1, C are all sums without x_1 and with x_6, and B are all other sums except s_{11} and s_{15}.

Maximum of the following points is taken:

- 7 p. for a complete solution;
- 2 p. for proving the bound for set A (like observation 1 from the solution);
- 2 p. for proving the bound for set C (like observation 2 from the solution);
- 4 p. for proving the bounds for two or three of the sets A, B, C.

Problem 5. *Marking scheme*

The points for the algorithm are summed with the points for the estimate.

- 4 p. for an algorithm that opens the safe in n attempts, covering all possible cases, with the proof that it works. Specifically, a proof of the fact that on some attempt the secret code will be entered (and not that we will just learn the code). *In the absence of such algorithm or proof, the maximum of the following points is taken:*
  - 2 p. for a correct algorithm without a proof;

3 p. for the estimate, i.e., proof that it is impossible to open the safe for sure in less than n attempts. *In the absence of such proof, the following advancement may be awarded with a point:*

  1 p. for a flawed proof of the estimate that contains an idea of 2-branching or an idea of quantity of information.

  In particular, 1 point is awarded if an incorrect proof of estimate for $n+1$ attempts could be easily corrected to a correct proof of estimate for n attempts.

  Another case when 1 point is awarded is sloppy 2-branching, when the third possible outcome (safe opens) is not accounted for (usually leading to the estimate of $n+1$).

The following advancements are not awarded with points:

0 p. just the answer;

0 p. just the answer with consideration of some small particular cases;

0 p. the algorithm with $n+1$ operations (including algorithms that *learn* the code in n operations, but do not enter it).

Problem 6. *Marking scheme*

Maximum of the following points is taken:

7 p. for a complete solution;

5 p. if problem is reduced to the lemma ($\angle AMD + \angle BMC + \angle AMC + \angle BMD > 2\pi$), and the lemma is stated, but not proven.

The following advancements are not awarded with points:

0 p. for the statement of the lemma (without proof);

0 p. for fori ncomplete computations (coordinates, vectors, trigonometry, etc.).