The 6th Olympiad of Metropolises

Mathematics

Solutions. Day 1

Problem 1. A positive integer is written on the board. Every minute Maxim adds to the number on the board one of its positive divisors, writes the result on the board and erases the previous number. However, it is forbidden for him to add the same number twice in a row. Prove that he can proceed in such a way that eventually a perfect square will appear on the board.

First solution. Let the number k be written on the board initially. We will assume that $k \geq 2$ (in the case of $k = 1$, a perfect square is already written on the board). Maxim can perform the following sequence of actions:

$$k \xrightarrow{k} 2k \xrightarrow{2k} 2 \cdot 2k \xrightarrow{2} 2 \cdot (2k+1) \xrightarrow{2k+1} 3 \cdot (2k+1) \xrightarrow{3}$$
$$3 \cdot (2k+2) \xrightarrow{2k+2} 4 \cdot (2k+2) \xrightarrow{4} 4 \cdot (2k+3) \xrightarrow{2k+3} \ldots$$
$$\ldots \xrightarrow{k^2-1} (k^2-2k+1)(k^2-1) \xrightarrow{k^2-2k+1} (k^2-2k+1) \cdot k^2\,,$$

where the added number is indicated above each arrow (it is obvious that any two numbers added in a row are different). It remains to note that $(k^2-2k+1) \cdot k^2 = \big(k(k-1)\big)^2$. □

Second solution. Let the number k be written on the board initially. We will prove that Maxim can get any number divisible by 6 and greater than $2k$. The statement of the problem easily follows from this, for example, he can obtain $(6k)^2$.

Suppose he got a number of the form $6n$ at some point. Let us show how he can obtain $6(n+1)$. This can be done either by sequence

$$6n \xrightarrow{3} (6n+3) \xrightarrow{1} (6n+4) \xrightarrow{2} (6n+6)\,,$$

or

$$6n \xrightarrow{2} (6n+2) \xrightarrow{1} (6n+3) \xrightarrow{3} (6n+6)\,,$$

depending on whether 2 or 3 was used to get $6n$. Thus, it remains to get some number divisible by 6. Let us consider several cases.

If $k = 6m+3$, Maxim can perform $(6m+3) \xrightarrow{3} (6m+6)$.

If $k = 6m+4$, Maxim can perform $(6m+4) \xrightarrow{2} (6m+6)$.

If $k = 6m+5$, Maxim can perform $(6m+5) \xrightarrow{1} (6m+6)$.

If $k = 6m+2$, Maxim can perform $(6m+2) \xrightarrow{1} (6m+3) \xrightarrow{3} (6m+6)$.

If $k = 6m + 1$, then for $m = 0$ he already has a perfect square $k = 1$, otherwise he can perform

$$(6m+1) \xrightarrow{6m+1} (12m+2) \xrightarrow{1} (12m+3) \xrightarrow{3} (12m+6). \qquad \square$$

Third solution. Suppose initially the number x was written on the board. Let $x = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}$, where $a_1 > a_2 > \ldots > a_k \geq 0$. Let us prove that if x has 0 in the second position in its binary notation (that is, $a_2 < a_1 - 1$), then we can get the number x^2.

To do this, at the first stage, Maxim can perform the operations

$$x \xrightarrow{x} 2x \xrightarrow{2x} 4x \xrightarrow{4x} \ldots \xrightarrow{2^{a_1-1}x} 2^{a_1}x.$$

At the second stage, he can perform the operations

$$2^{a_1}x \xrightarrow{2^{a_2}x} 2^{a_1}x + 2^{a_2}x \xrightarrow{2^{a_3}x} \ldots \xrightarrow{2^{a_k}x} 2^{a_1}x + 2^{a_2}x + \ldots + 2^{a_k}x = x^2.$$

At the first stage, each time the number added is twice the number of the previous step, so any two numbers added in a row are different. At the second stage, different powers of two multiplied by x are added, so any two consecutive numbers are also different. At the junction of the stages, the numbers $2^{a_1-1}x$ and $2^{a_2}x$ are added subsequently, and they are also different, since $a_2 < a_1 - 1$.

It remains to deal with the case when x has 1 in the second position of its binary notation (that is, $a_2 = a_1 - 1$). Note that it is enough for Maxim to get from x any number with 0 as the second digit and then repeat the previous algorithm (the same numbers will not be added twice in a row, since the first step of the above algorithm is doubling the number). To do this, Maxim will obtain the number $12x$ with

$$x \xrightarrow{x} 2x \xrightarrow{2x} 4x \xrightarrow{x} 5x \xrightarrow{5x} 10x \xrightarrow{2x} 12x.$$

Let us check that $12x$ in the second position in binary notation will have 0. Indeed, for $x = 2^{a_1} + 2^{a_1-1} + S$, where $0 \leq S < 2^{a_1-1}$, we have

$$\begin{aligned} 2^{a_1+4} = 12 \cdot 2^{a_1} + 8 \cdot 2^{a_1-1} < 12x = 12 \cdot (3 \cdot 2^{a_1-1} + S) = 36 \cdot 2^{a_1-1} + 12 \cdot S < \\ < 36 \cdot 2^{a_1-1} + 12 \cdot 2^{a_1-1} = 2^{a_1+4} + 2^{a_1+3}, \end{aligned}$$

which means that the number $12x$ has 0 in the second position of its binary notation. $\square$

Fourth solution. Let Maxim at each step increase the largest power of 2 that the number on the board is divisible by. Namely, to the number $x = 2^k \cdot a$, where a is odd, Maxim will add its divisor 2^k. Then

$$x + 2^k = 2^k(a+1) = 2^{k+s} \cdot \frac{a+1}{2^s},$$

where the number $\frac{a+1}{2^s}$ is odd. Note that $s \geq 1$, therefore $\frac{a+1}{2^s} < a$ if $a > 1$. Thus, the largest odd divisor of the number on the board will always decrease until it becomes equal to 1, that is, sooner or later the number on the board will become some power of two, a number of the form 2^n. If n is even, then Maxim has achieved what he wants. If n is odd, Maxim can add 2^n to this number, after which he will get a perfect square.

In the course of this algorithm, different numbers are added each time, since the largest power of 2 by which the number on the board is divisible increases. □

Problem 2. Points P and Q are chosen on the side BC of triangle ABC so that P lies between B and Q. The rays AP and AQ divide the angle BAC into three equal parts. It is known that the triangle APQ is acute-angled. Denote by B_1, P_1, Q_1, C_1 the projections of points B, P, Q, C onto the lines AP, AQ, AP, AQ, respectively. Prove that lines B_1P_1 and C_1Q_1 meet on line BC.

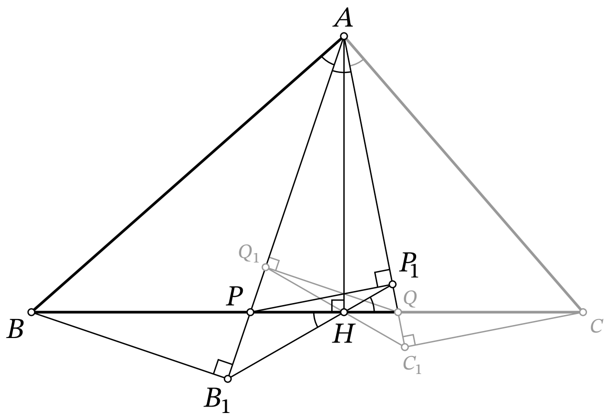

Figure 1: for the solution of problem 2

Solution. Let AH be the altitude of the triangle ABC (fig. 1). The points A, B, B_1 and H lie on the circle with diameter AB and the points A, P, P_1 and H lie on the circle with diameter AP. Hence

$$\angle BHB_1 = \angle BAB_1 = \angle PAP_1 = \angle QHP_1,$$

so the lines HB_1 and HP_1 coincide. Thus the line B_1P_1 passes through H. Similarly, the line C_1Q_1 passes through H. □

Another solution. Let X and Y be the intersection points of the line BC with the lines B_1P_1 and C_1Q_1 respectively. By Menelaus's theorem

$$\frac{PX}{XQ} \cdot \frac{QP_1}{P_1A} \cdot \frac{AB_1}{B_1P} = 1 \qquad \text{and} \qquad \frac{QY}{YP} \cdot \frac{PQ_1}{Q_1A} \cdot \frac{AC_1}{C_1Q} = 1\,.$$

Multiplying these equalities, we obtain

$$\frac{XQ}{PX} \cdot \frac{YP}{QY} = \frac{QP_1}{P_1A} \cdot \frac{AB_1}{B_1P} \cdot \frac{PQ_1}{Q_1A} \cdot \frac{AC_1}{C_1Q} = \left(\frac{QP_1}{C_1Q} \cdot \frac{AC_1}{P_1A}\right) \cdot \left(\frac{PQ_1}{B_1P} \cdot \frac{AB_1}{Q_1A}\right). \tag{*}$$

Note that $QP_1 : C_1Q = QP : CQ = AP : AC = P_1A : AC_1$ and $PQ_1 : B_1P = PQ : BP = AQ : AB = Q_1A : AB_1$. Hence the right-hand side of (*) equals 1. So $QX : XP = QY : YP$ and the points X and Y coincide. □

Problem 3. Let $a_1, a_2, \dots, a_n$ ($n \geq 2$) be nonnegative real numbers whose sum is $\frac{n}{2}$. For every $i = 1, \dots, n$ define

$$b_i = a_i + a_i a_{i+1} + a_i a_{i+1} a_{i+2} + \cdots + a_i a_{i+1} \cdots a_{i+n-2} + 2 a_i a_{i+1} \cdots a_{i+n-1}\,,$$

where $a_{j+n} = a_j$ for every j. Prove that $b_i \geq 1$ holds for at least one index i.

First solution. All indices in the solution are considered modulo n.

Lemma. There exists an index i such that if we denote $x_1 = a_{i+1}$, $x_2 = a_{i+2}$, etc., then

$$x_1 + x_2 + \ldots + x_j \geq \frac{j}{2} \quad \text{for every } j = 1, 2, \dots, n\,. \tag{♤}$$

Proof of the lemma. Let us choose i so that the value of $a_1 + a_2 + \ldots + a_i - \frac{i}{2}$ is the smallest possible (since $a_1 + a_2 + \ldots + a_n = \frac{n}{2}$, such values will be the same for i and $i+n$). Then for any j we have

$$a_{i+1} + a_{i+2} + \ldots a_{i+j} = \frac{j}{2} + \left(a_1 + a_2 + \ldots + a_{i+j} - \frac{i+j}{2}\right) - \left(a_1 + a_2 + \ldots + a_i - \frac{i}{2}\right) \geq \frac{j}{2}\,.$$

Lemma is proven.

Denoting x_j in accordance with the lemma, we prove by induction on k that if (♤) holds for $j \leq k$, then

$$x_1 + x_1 x_2 + \ldots + x_1 x_2 \cdots x_{k-1} + 2 x_1 x_2 \cdots x_k \geq 1\,. \tag{♡}$$

For $k = n$ this will give the required $b_{i+1} \geq 1$.

For $k = 1$, the inequality $2x_1 \geq 1$ is obvious; suppose that (♡) holds for $k-1$, where $k > 1$. The induction step will follow from (♡) applied to $k-1$ numbers x_1, ..., x_{k-2}, $\frac{1}{2}x_{k-1} + x_{k-1}x_k$, so it suffices to check that this sequence satisfies conditions (♤).

We denote $x_1 + \cdots + x_{k-2} = \frac{k-1}{2} - s$, where $s \leq \frac{1}{2}$; we need to prove that $\frac{1}{2}x_{k-1} + x_{k-1}x_k \geq s$. For $s \leq 0$ or $x_{k-1} > 1$, this is obvious, so let us consider the case $0 \leq s \leq \frac{1}{2}$ and $0 \leq x_{k-1} \leq 1$. Since we know $x_{k-1} \geq s$ and $x_{k-1} + x_k \geq s + \frac{1}{2}$ from conditions (♤) for $j = k-1$ and k, then

$$\tfrac{1}{2}x_{k-1} + x_{k-1}x_k = x_{k-1}(x_{k-1} + x_k - \tfrac{1}{2}) + (1 - x_{k-1})x_{k-1} \geq x_{k-1}s + (1 - x_{k-1})s = s\,,$$

as required. □

Second solution. Assume that $b_i < 1$ for all i. Then also $a_i \leq b_i < 1$ for all i. Denote $A = a_1 a_2 \ldots a_n$. We have

$$b_{i-1} = a_{i-1} + a_{i-1} b_i + A - 2A a_{i-1}$$

(all indices are considered modulo n). Sum this up for $i = 1, 2, \dots, n$ and substitute $\sum_i a_i = \frac{n}{2}$ to get

$$\sum_i b_i = \frac{n}{2} + \sum_i b_i a_{i-1} + nA - nA \quad \Rightarrow$$

$$\frac{n}{2} = \sum_i b_i(1 - a_{i-1}) < \sum_i (1 - a_{i-1}) = \frac{n}{2}\,,$$

a contradiction. □