6th Olympiad of Metropolises

Mathematics · Day 1

Problem 1. A positive integer is written on the board. Every minute Maxim adds to the number on the board one of its positive divisors, writes the result on the board and erases the previous number. However, it is forbidden for him to add the same number twice in a row. Prove that he can proceed in such a way that eventually a perfect square will appear on the board.

Problem 2. Points P and Q are chosen on the side BC of triangle ABC so that P lies between B and Q. The rays AP and AQ divide the angle BAC into three equal parts. It is known that the triangle APQ is acute-angled. Denote by B_1, P_1, Q_1, C_1 the projections of points B, P, Q, C onto the lines AP, AQ, AP, AQ, respectively. Prove that lines B_1P_1 and C_1Q_1 meet on line BC.

Problem 3. Let $a_1, a_2, \ldots, a_n$ $(n \geq 2)$ be nonnegative real numbers whose sum is $\frac{n}{2}$. For every $i = 1, \ldots, n$ define

$$b_i = a_i + a_i a_{i+1} + a_i a_{i+1} a_{i+2} + \cdots + a_i a_{i+1} \cdots a_{i+n-2} + 2a_i a_{i+1} \cdots a_{i+n-1},$$

where $a_{j+n} = a_j$ for every j. Prove that $b_i \geq 1$ holds for at least one index i.