The 5th Olympiad of Metropolises

Mathematics

Marking scheme

Day 1

Problem 1. In a triangle ABC with a right angle at C, the angle bisector AL (where L is on segment BC) intersects the altitude CH at point K. The bisector of angle BCH intersects segment AB at point M. Prove that $CK = ML$. *(Alexey Doledenok)*

Marking scheme

Maximum of the following points is taken:

7 p. for a complete solution;

2 p. for proving $CK = CL$;

2 p. for proving $CL = ML$.

Problem 2. Does there exist a positive integer n such that all its digits (in the decimal system) are greater than 5, while all the digits of n^2 are less than 5? *(Nazar Agakhanov)*

Marking scheme

Maximum of the following points is taken:

7 p. for a complete solution;

6 p. for an otherwise complete solution that contains minor arithmetic flaws, like computational errors in proofs of some rough estimates that are clearly true anyway.

4 p. for a correct proof of the key inequality $\underbrace{444...4}_{2k} < \underbrace{666...67}_{k}^2$, if the further logical part is absent or incorrect (for example, the total number of digits is not estimated properly or the solution is based on vague manipulations with digits).

3 p. for a correct and precise statement of the key inequality $\underbrace{444...4}_{2k} < \underbrace{666...67}_{k}^2$ and clear further logical part, but without a formal proof of this inequality.

1 p. for an attempt to prove an inequality that is similar in idea to the inequality $\underbrace{444...4}_{2k} < \underbrace{666...67}_{k}^2$.

The following advancement is not awarded with points:

0 p. for noticing that the first digit must be 6 and the last digit must be 8 or 9. The same goes for every specific finite prefix or suffix of the number (for example when all the possibilities for the first or the last 10 digits of the number are described).

Problem 3. Let $n > 1$ be a given integer. The Mint issues coins of n different values a_1, a_2, ..., a_n, where each a_i is a positive integer (the number of coins of each value is unlimited). A set of values $\{a_1, a_2, \dots, a_n\}$ is called *lucky,* if the sum $a_1 + a_2 + \dots + a_n$ can be collected in a unique way (namely, by taking one coin of each value).

(a) Prove that there exists a lucky set of values $\{a_1, a_2, \dots, a_n\}$ with

$$a_1 + a_2 + \dots + a_n < n2^n.$$

(b) Prove that every lucky set of values $\{a_1, a_2, \dots, a_n\}$ satisfies

$$a_1 + a_2 + \dots + a_n > n2^{n-1}.$$

(Ilya Bogdanov)

Marking scheme

The sum of the following points is taken:

- 3 p. for proving part (a). In the absence of a complete proof, the following advancement is awarded with points:
  - 1 p. for providing an example of lucky set satisfying (per the jury's knowledge) the condition of part (a) for each n, but without correct proof that the example is indeed valid.
- 4 p. for proving part (b).

Day 2

Problem 4. Positive numbers a, b and c satisfy $a^2 = b^2 + bc$ and $b^2 = c^2 + ac$. Prove that $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. *(Vladimir Bragin)*

Marking scheme

Any complete solution is awarded 7 points.

Problem 5. There is an empty table with 2^{100} rows and 100 columns. Alice and Eva take turns filling the empty cells of the first row of the table, Alice plays first. In each move, Alice chooses an empty cell and puts a cross in it; Eva in each move chooses an empty cell and puts a zero. When no empty cells remain in the first row, the players move on to the second row, and so on (in each new row Alice plays first).

The game ends when all the rows are filled. Alice wants to make as many different rows in the table as possible, while Eva wants to make as few as possible. How many different rows will be there in the table if both follow their best strategies? *(Denis Afrizonov)*

Marking scheme

The sum of the following points is taken:

5 p. for proving that Alice can achieve $\geq 2^{50}$ different rows.

2 p. for proving that Eva can achieve $\leq 2^{50}$ different rows.

The following advancement is not awarded with points:

0 p. for the correct answer without further reasoning.

Problem 6. Consider a convex pentagon $ABCDE$. Let A_1, B_1, C_1, D_1, E_1 be the intersection points of the pairs of diagonals BD and CE, CE and DA, DA and EB, EB and AC, AC and BD, respectively. Prove that if four of the five quadrilaterals AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E, EA_1E_1A are cyclic, then the fifth one is also cyclic. *(Nairi Sedrakyan, Yuliy Tikhonov)*

Marking scheme

Maximum of the following points is taken:

7 p. for a complete solution;

3 p. for reducing the problem to the relation

$$\frac{AD_1}{BD_1} \cdot \frac{BE_1}{CE_1} \cdot \frac{CA_1}{DA_1} \cdot \frac{DB_1}{EB_1} \cdot \frac{EC_1}{AC_1} = 1\,. \tag{*}$$

2 p. for showing

$$\sin\angle BAC : \sin\angle DEC = \sin\angle EAD : \sin\angle BEA\,. \tag{**}$$

(Can also be expressed as a double ratio of sines that equals 1.)

The following advancements are not awarded with points:

0 p. for incomplete computations (coordinates, complex numbers, vectors, trigonometry, etc.)

0 p. for just angle chasing (in particular, for proving $\angle ABE = \angle CBD$, etc.);

0 p. for applying sine law, Ceva theorem, etc., unless it leads to (**);

0 p. for applying power-of-a-point theorem, unless it leads to (*).