2nd Olympiad of Metropolises

Day 2

Problem 1. Find the largest positive integer N for which one can choose N distinct numbers from the set $\{1, 2, 3, \ldots, 100\}$ such that neither the sum nor the product of any two different chosen numbers is divisible by 100. *(Mikhail Evdokimov)*

Answer: 45.

Example. Let us choose all numbers from 1 to 49 except 20, 25, 30, and 40. The sum of any two chosen numbers does not exceed 97, and thus cannot be a multiple of 100. Moreover, since none of these numbers is a multiple of 25 and the only multiple of 10 is 10 itself, the product of any two chosen numbers cannot be a multiple of 100. Therefore this set of numbers satisfies the condition.

Estimate. Consider the following 46 groups of numbers. The first group contains all multiples of 10 (i. e. 10, 20, …, 100), whereas the other 45 groups consist of the pairs of the remaining numbers summing up to 100 (i. e. 1 and 99, 2 and 98, …, 9 and 91, 11 and 89, …, 49 and 51). Since the product of any two numbers from the first group is divisible by 100 and the sum of the two numbers in each pair is 100, we can choose at most one number from each group. Moreover, we must not choose numbers from both pairs (4, 96) and (25, 75), because the product of these two numbers would be a multiple of 100. Hence, we can choose at most 45 numbers. □

Problem 2. Let x and y be positive integers greater than 1 such that

$$[x+2, y+2] - [x+1, y+1] = [x+1, y+1] - [x, y] .$$

Prove that one of the two numbers x and y divides the other.

(Here $[a, b]$ denotes the least common multiple of a and b.) *(Dušan Djukić)*

The case $x = y$ is trivial. Assume without loss of generality that $x < y$.

Note that for any positive integers m and n it holds that $[m, n] = cn$, where $c = m/(m, n)$ is a positive divisor of m (as usual, (m, n) denotes the greatest common divisor of m and n). Therefore, $[x, y] + [x+2, y+2] = 2[x+1, y+1]$ implies

$$ay + c(y+2) = 2(y+1)b , \tag{1}$$

where

$$a = \frac{x}{(x, y)} , \quad b = \frac{x+1}{(x+1, y+1)} , \quad c = \frac{x+2}{(x+2, y+2)} . \tag{2}$$

In particular,

$$a \mid x , \quad b \mid (x+1) , \tag{3}$$

$$1 \le a \le x < y+1\,, \quad 1 \le c \le x+2 \le y+1\,. \tag{4}$$

It follows from (1) that $c-a=(2b-a-c)(y+1)$, so $(y+1) \mid (c-a)$. But (4) implies $|c-a|<y+1$, so $c-a=0$, i. e., $a=c$. Now (1) also gives us $b=a$, and from (3) we deduce that $a \mid x$ and $a \mid (x+1)$, so $a=1$. Finally, now (2) yields $x=(x,y)$, and hence $x \mid y$. □

Problem 3. Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF, and EFA, respectively. Let ℓ_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines ℓ_{BC}, ℓ_{CD}, ℓ_{DE}, ℓ_{EF}, and ℓ_{FA} are analogously defined. Let A_1 be the intersection point of the lines ℓ_{FA} and ℓ_{AB}; B_1 be the intersection point of the lines ℓ_{AB} and ℓ_{BC}; points C_1, D_1, E_1, and F_1 are analogously defined.

Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point. *(Nairi Sedrakyan)*

Solution 1.

We will prove that the hexagon $A_1B_1C_1D_1E_1F_1$ is centrally symmetric and therefore the main diagonals A_1D_1, B_1E_1 and C_1F_1 pass through the symmetry center of the hexagon.

We denote the centers of ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F by I_A, I_B, I_C, I_D, I_E, and I_F, respectively.

Claim 1. $\ell_{AB} \parallel CF \parallel \ell_{DE}$, $\ell_{BC} \parallel AD \parallel \ell_{EF}$, and $\ell_{CD} \parallel BE \parallel \ell_{FA}$.

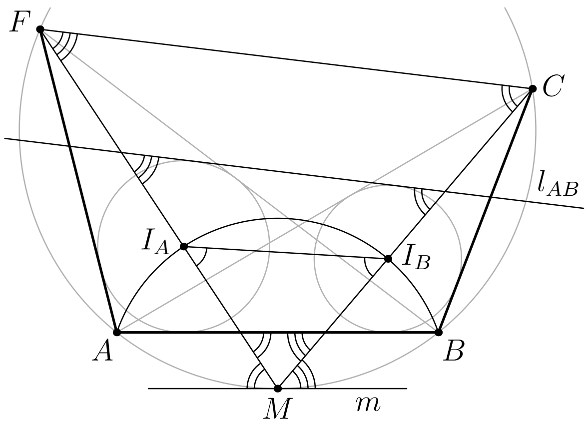

Figure 1: solution 1 of problem 6.

Proof. By the symmetry it suffices to prove that $\ell_{AB} \parallel CF$. Let M be the midpoint of the arc AB of the circumcircle not containing C and F, and let

m be the tangent to the circumcircle at M, which is parallel to AB (Fig. 1). Then we have

$$\angle(FM, CF) = \angle(m, CM) = \angle(AB, CM) . \tag{5}$$

It is well known that the incenter I_A of the triangle FAB satisfies $MI_A = MA = MB$; similarly, $MI_B = MA = MB$, so $MI_A = MI_B$ and therefore $\angle(I_AI_B, CM) = \angle(FM, I_AI_B)$. The lines AB and ℓ_{AB} are symmetric about the line I_AI_B, so

$$\begin{aligned}\angle(AB, CM) &= \angle(AB, I_AI_B) + \angle(I_AI_B, CM)\\ &= \angle(I_AI_B, \ell_{AB}) + \angle(FM, I_AI_B) = \angle(FM, \ell_{AB}) .\end{aligned}$$

This equation combined with (5) yields $\angle(FM, CF) = \angle(FM, \ell_{AB})$, so indeed $CF \parallel \ell_{AB}$. □

Claim 2. $A_1B_1 + C_1D_1 + E_1F_1 = B_1C_1 + D_1E_1 + F_1A_1$.

Proof. Let T_A, U_A, V_A, and W_A be the points where ω_A touches the lines AB, FA, ℓ_{AB}, and ℓ_{FA}, respectively, and define the points $T_B, \dots, W_F$ analogously (Fig. 2). Since the hexagon $ABCDEF$ is tangential, we have

$$AB + CD + EF = BC + DE + FA. \tag{6}$$

Furthermore we have

$$AT_A = AU_A, \dots, FT_F = FU_F \quad \text{and} \quad A_1V_A = A_1W_A, \dots, F_1V_F = F_1W_F, \tag{7}$$

because these pairs of segments are tangents drawn to the circles $\omega_A, \dots, \omega_F$.

Finally, from the symmetry about the lines $I_AI_B, \dots, I_FI_A$, we can see that

$$T_AU_B = V_AW_B, \dots, T_FU_A = V_FW_A. \tag{8}$$

By combining (7) and (8),

$$\begin{aligned}A_1B_1 &= V_AW_B - A_1V_A - B_1W_B = T_AU_B - A_1V_A - B_1W_B\\ &= (AB - AT_A - BU_B) - A_1V_A - B_1W_B\\ &= AB - AT_A - BT_B - A_1V_A - B_1V_B .\end{aligned}$$

Analogously,

$$\begin{aligned}B_1C_1 &= BC - BT_B - CT_C - B_1V_B - C_1V_C ,\\ &\dots\\ F_1A_1 &= FA - FT_F - AT_A - F_1V_F - A_1V_A .\end{aligned}$$

Now the Claim can be achieved by plugging these formulas into (6) and cancelling identical terms. □

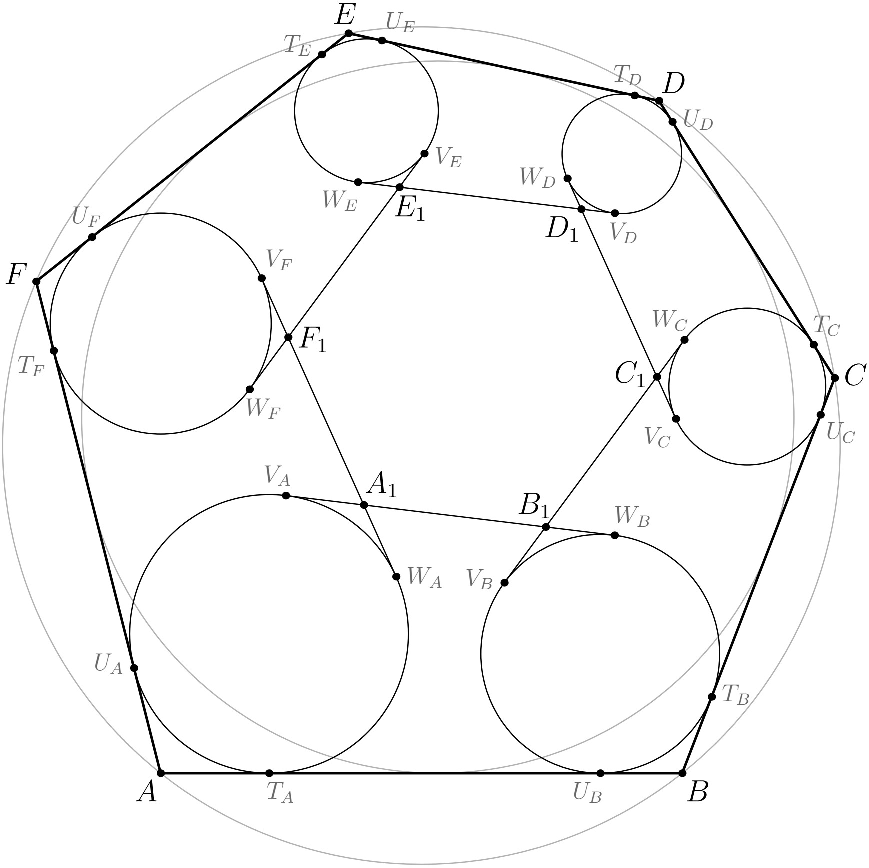

Figure 2: solution 1 of problem 6.

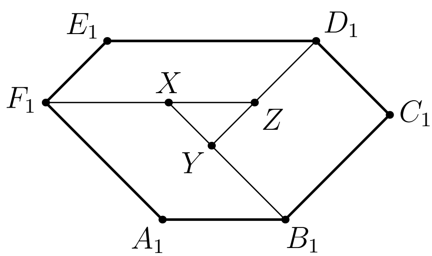

Figure 3: solution 1 of problem 6.

Claim 3. $\overrightarrow{A_1B_1} = \overrightarrow{E_1D_1}$, $\overrightarrow{B_1C_1} = \overrightarrow{F_1E_1}$, and $\overrightarrow{C_1D_1} = \overrightarrow{A_1F_1}$.

Proof. Let X, Y, and Z be those points for which the quadrilaterals $F_1A_1B_1X$,

$B_1C_1D_1Y$, and $D_1E_1F_1Z$ are parallelograms (Fig. 3). By Claim 1 we have $F_1X \parallel A_1B_1 \parallel E_1D_1 \parallel F_1Z$, so the points F_1, X, Z are collinear; it can be seen similarly that B_1, X, Y are collinear and D_1, Y, Z are also collinear. We will show that the points X, Y, Z coincide.

The points X, Y, Z either coincide or form a triangle. Suppose that XYZ is a triangle with the same orientation as the hexagon $A_1B_1C_1D_1E_1F_1$. Then

$$\begin{aligned} F_1A_1 + B_1C_1 + D_1E_1 &= XB_1 + YD_1 + ZF_1 \\ &> YB_1 + ZD_1 + XF_1 = C_1D_1 + E_1F_1 + A_1B_1\,, \end{aligned}$$

contradicting Claim 2.

If XYZ is a triangle with the opposite orientation from the hexagon, we get a contradiction in the same way, for then we have $F_1A_1 + B_1C_1 + D_1E_1 < C_1D_1 + E_1F_1 + A_1B_1$, □

Claim 3 shows that the hexagon $A_1B_1C_1D_1E_1F_1$ is indeed centrally symmetric, as required. □

Solution 2.

We present an alternative finishing of the solution after Claim 1. We denote by I the incenter of the hexagon $ABCDEF$.

Claim 4. $II_A \cdot IA = II_B \cdot IB = II_C \cdot IC = II_D \cdot ID = II_E \cdot IE = II_F \cdot IF$.

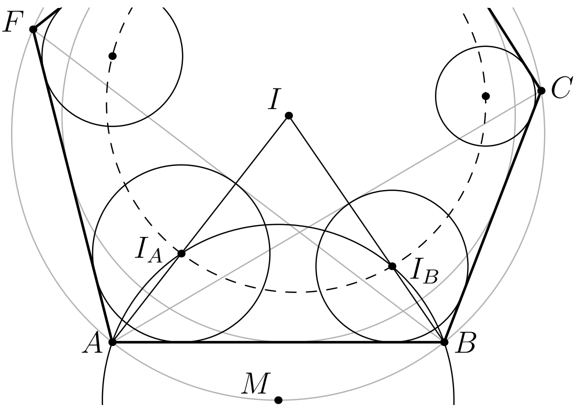

Figure 4: solution 2 of problem 6.

Proof. Again, let M be the midpoint of the arc AB of the circumcircle not containing C. As was already mentioned in the proof of Claim 1, the points A, B, I_A, and I_B lie on a circle centered at M (Fig. 4). Since AI_A and BI_B are bisectors of the angles FAB and ABC, they meet at I. Thus $II_A \cdot IA =$

$II_B \cdot IB$ is the power of I with respect to the circle mentioned above. The other equalities can be proved in a similar way. $\square$

Claim 4 implies that the inversion f with center I and radius $\rho = \sqrt{II_A \cdot IA}$ maps the points $A, \dots, F$ to $I_A, \dots, I_F$, respectively. Therefore, the hexagon $I_A I_B I_C I_D I_E I_F$ is cyclic.

It is well-known that the main diagonals of a cyclic hexagon $X_1X_2X_3X_4X_5X_6$ are concurrent if and only if $X_1X_2 \cdot X_3X_4 \cdot X_5X_6 = X_4X_5 \cdot X_6X_1 \cdot X_2X_3$ (this fact follows from the trigonometric form of Ceva's theorem for triangle $X_1X_3X_5$). Thus, since $ABCDEF$ is both cyclic and tangential, by the Brianchon theorem we obtain

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA. \tag{9}$$

On the other hand, by means of the inversion f we obtain

$$AB = I_A I_B \cdot \frac{IA \cdot IB}{\rho^2}, \quad \dots, \quad FA = I_F I_A \cdot \frac{IF \cdot IA}{\rho^2}.$$

Plugging this into (9) we obtain that

$$I_A I_B \cdot I_C I_D \cdot I_E I_F = I_B I_C \cdot I_D I_E \cdot I_F I_A.$$

which in turn means that the diagonals $I_A I_D$, $I_B I_E$, and $I_C I_F$ are concurrent.

Claim 5. Line $I_A I_D$ is the angle bisector of the angles $F_1A_1B_1$ and $C_1D_1E_1$; similarly, lines $I_B I_E$ and $I_C I_F$ bisect the angles $A_1B_1C_1$, $D_1E_1F_1$, $B_1C_1D_1$, and $E_1F_1A_1$.

Proof. Let the angle bisectors of $\angle FAB$ and $\angle CDE$ meet the circumcircle of $ABCDEF$ again at N_A and N_D, respectively (Fig. 5); notice that

$$\angle(N_A N_D, BE) = \frac{1}{2}(\widehat{N_D E} + \widehat{N_A B}) = \\ = \frac{1}{2}(\widehat{CN_D} + \widehat{FN_A}) = \angle(CF, N_A N_D).$$

Thus $N_A N_D$ is parallel to the bisector of the angle between BE and CF (containing A). By Claim 1, $N_A N_D$ is also parallel to the bisectors of angles $F_1A_1B_1$ and $C_1D_1E_1$.

Since the points A, D, N_A, and N_D are concyclic, lines $N_A N_D$ and AD are anti-parallel with respect to angle AID. On the other hand, by Claim 4 the points A, D, I_A, and I_D are concyclic or collinear, which means that lines AD and $I_A I_D$ are antiparallel with respect to the same angle. The two observations yield $N_A N_D \parallel I_A I_D$.

Thus, the two angle bisectors are parallel to $I_A I_D$ and pass through I_A and I_D, respectively. This proves the first part of the Claim; the proofs of the other parts are similar. $\square$

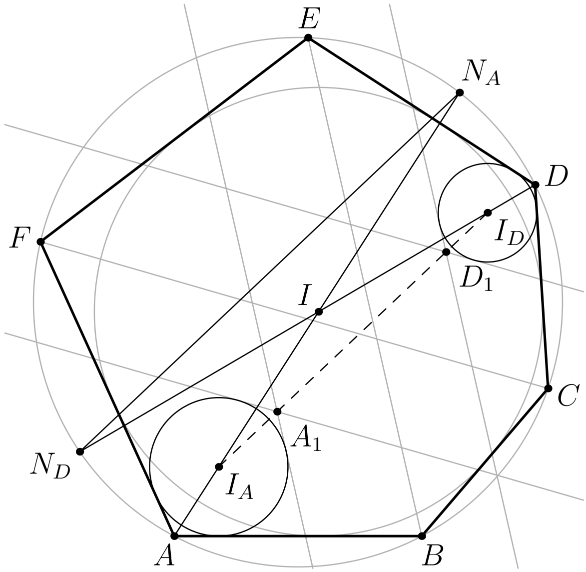

Figure 5: solution 2 of problem 6.

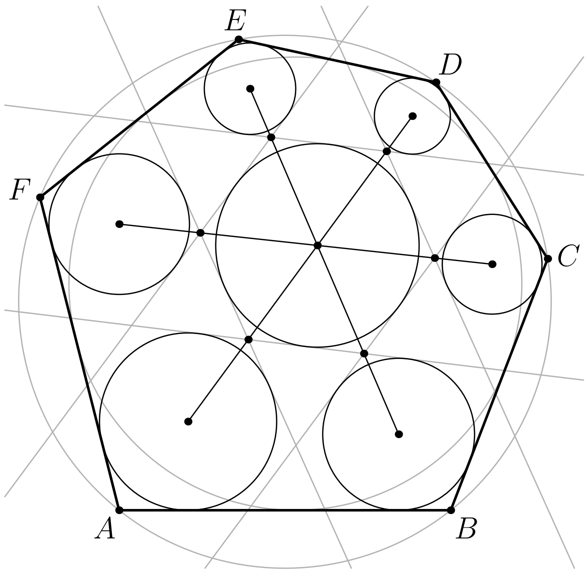

Figure 6: solution 2 of problem 6.

Claim 5, along with the above discussion, shows that the diagonals $I_A A_1 D_1 I_D$, $I_B B_1 E_1 I_E$, and $I_C C_1 F_1 I_F$ are concurrent and bisect the angles of the hexagon $A_1 B_1 C_1 D_1 E_1 F_1$ (Fig. 6). This shows another interesting fact: namely, this hexagon is tangential and the diagonals connecting opposite points pass through its incenter. □

Solution 3.

Using Claim 4 from the previous solution, we shall prove a more general statement:

Theorem. Two circles Ω and ω are given in the plane such that ω lies inside Ω. Consider an arbitrary broken line $ABCD$ inscribed in Ω whose segments AB, BC and CD are tangent to ω. Let I_B and I_C be the incenters of the triangles ABC and BCD, respectively. Let ℓ be the line symmetric to the line BC about the line I_BI_C. Then ℓ touches a fixed circle independent of the choice of the broken line $ABCD$.

Lemma. Let Ω and ω be two circles with centers at O and I, respectively, such that ω lies inside Ω. Consider an arbitrary chord BC of the circle Ω tangent to ω. Then the circumcenter X of the triangle BIC lies on a fixed circle Γ centered at O, independent of the choice of BC.

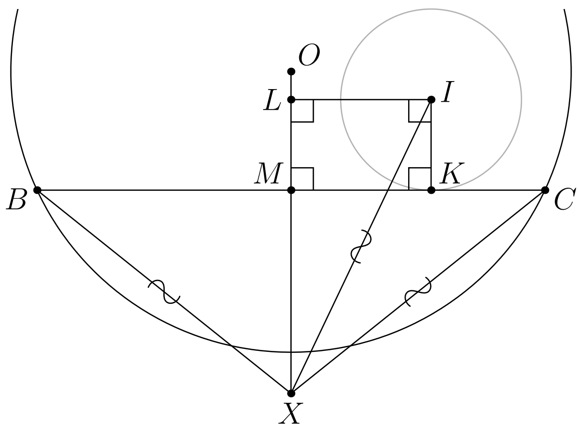

Figure 7: problem 6, third solution.

Proof of the lemma. Let K and L be the projections of I on the lines BC and OX, respectively, and let M be the midpoint of BC (Fig. 7). Obviously, $M \in OX$ and $MKIL$ is a rectangle. We have

$$\begin{aligned} OB^2 - OI^2 &= (OB^2 - XB^2) - (XI^2 - OI^2) \\ &= (OM^2 - XM^2) - (XL^2 - OL^2) \\ &= (OM + XM)(OM - XM) - (XL + OL)(XL - OL) \\ &= 2OX \cdot LM = 2OX \cdot IK. \end{aligned}$$

The lengths of segments OB, OI and IK do not depend on the chord BC, so neither does the length of OX. □

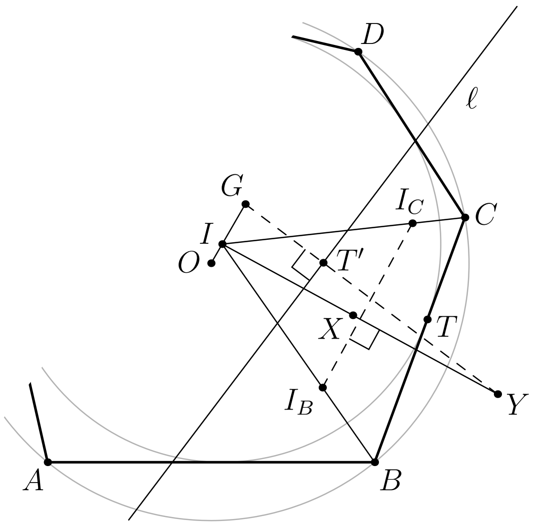

Figure 8: solution 3 of problem 6.

Proof of the theorem. Point X and circle Γ are defined as above. Consider the inversion f centered at I that maps I_B and I_C to B and C, respectively. (Its existence and independence from the broken line follows from the corollary of Claim 4).

Denote $\gamma = f(\Gamma)$. Let G be the center of γ and let T be the tangency point of ω and BC (Fig. 8). It is well-known that f maps the circumcenter X of the triangle BIC to the point Y symmetric to I about the line I_BI_C. We shall prove that the distance from G to the variable line ℓ is constant.

We show first that $GY \perp \ell$. Points B and C are symmetric about OX, so $OX \perp BC$. Also, since $IT \perp BC$, the lines IX and IT are symmetric about the bisector of angle BIC, and the lines BC and I_BI_C are anti-parallel with respect to the angle BIC, we have $IX \perp I_BI_C$. Now from $OX \perp BC$ and $IX \perp I_BI_C$ we deduce that $\angle(OX, IX) = \angle(BC, I_BI_C)$. By properties of inversion one can obtain $\angle(OX, IX) = \angle(IY, GY)$ (homothety of γ and Γ is useful here). By the symmetry in I_BI_C we have $\angle(BC, I_BI_C) = \angle(I_BI_C, \ell)$. Thus $\angle(IY, GY) = (I_BI_C, \ell)$, so $IY \perp IB_1IC_1$ implies that $GY \perp \ell$.

Let T' be the reflection of T in I_BI_C. The segments IT and YT' are symmetric about the line I_BI_C and so are the lines BC and ℓ. Since $YT' \perp \ell$ it follows that G lies on YT'. Finally, the distance between G and ℓ is equal to $GY - YT' =$

$GY - IT = R(\gamma) - R(\omega)$, which is constant. (Here $R(s)$ denotes the radius of a circle s.) □

The problem statement easily follows from this theorem. Observe that all lines ℓ_{AB}, ℓ_{BC}, ℓ_{CD}, ℓ_{DE}, ℓ_{EF}, ℓ_{FA} are tangent to the same circle, which implies that the hexagon $A_1B_1C_1D_1E_1F_1$ is tangential. By the Brianchon theorem its diagonals are concurrent. □