The 3rd Olympiad of Metropolises

Day 1. Problems

Problem 1. Solve the system of equations in real numbers:

$$\begin{cases} (x-1)(y-1)(z-1) = xyz - 1\,, \\ (x-2)(y-2)(z-2) = xyz - 2\,. \end{cases}$$

(Vladimir Bragin)

Problem 2. A convex quadrilateral $ABCD$ is circumscribed about a circle ω. Let PQ be the diameter of ω perpendicular to AC. Suppose lines BP and DQ intersect at point X, and lines BQ and DP intersect at point Y. Show that the points X and Y lie on the line AC. *(Géza Kós)*

Problem 3. Let k be a positive integer such that $p = 8k+5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1 \leqslant i < j \leqslant 2k+1} \left(r_i^4 + r_j^4\right)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

(Two integers are congruent modulo p if p divides their difference.) *(Fedor Petrov)*