Day 1. Solutions

Problem 1. Solve the system of equations in real numbers:

$$\begin{cases} (x-1)(y-1)(z-1) = xyz - 1\,, \\ (x-2)(y-2)(z-2) = xyz - 2\,. \end{cases}$$

(Vladimir Bragin)

Answer: $x = 1$, $y = 1$, $z = 1$.

Solution 1. By expanding the parentheses and reducing common terms we obtain

$$\begin{cases} -(xy + yz + zx) + (x + y + z) = 0\,, \\ -2(xy + yz + zx) + 4(x + y + z) = 6\,. \end{cases}$$

From the first equation we can conclude that $xy+yz+zx = x+y+z$. By substituting this into the second equation, we obtain that $x + y + z = 3$. We now have to solve the system

$$\begin{cases} x + y + z = 3\,, \\ xy + yz + zx = 3\,. \end{cases} \tag{1}$$

If we square the first equation, we get $x^2 + y^2 + z^2 + 2(xy + yz + zx) = 9$. Hence $x^2 + y^2 + z^2 = 3 = xy + yz + zx$.

We will prove that if $x^2 + y^2 + z^2 = xy + yz + zx$, than $x = y = z$:

$$\begin{gathered} x^2 + y^2 + z^2 = xy + yz + zx \quad \iff \\ 2x^2 + 2y^2 + 2z^2 = 2xy + 2yz + 2zx \quad \iff \\ x^2 - 2xy + y^2 + x^2 - 2xz + z^2 + y^2 - 2yz + z^2 = 0 \quad \iff \\ (x-y)^2 + (x-z)^2 + (y-z)^2 = 0\,. \end{gathered}$$

The sum of three squares is 0, so all of them are zeroes, which implies $x = y = z$. That means $x = y = z = 1$. □

Solution 1'. We will show one more way to solve the system (1). Express $z = 3-x-y$ from first equation and substitute it into the second one:

$$\begin{gathered} xy + (y + x)(3 - x - y) = 3 \quad \iff \\ xy + 3x + 3y - 2xy - x^2 - y^2 = 3 \quad \iff \\ x^2 + y^2 + xy - 3x - 3y + 3 = 0 \quad \iff \\ x^2 + x(y - 3) + y^2 - 3y + 3 = 0\,. \end{gathered}$$

Let us solve it as a quadratic equation over variable x:

$$\begin{aligned}x &= \frac{(3-y) \pm \sqrt{(y-3)^2 - 4(y^2-3y+3)}}{2} = \\ &= \frac{(3-y) \pm \sqrt{y^2 - 6y + 9 - 4y^2 + 12y - 12}}{2} = \\ &= \frac{(3-y) \pm \sqrt{-3y^2 + 6y - 3}}{2} = \\ &= \frac{(3-y) \pm \sqrt{-3(y-1)^2}}{2}\,.\end{aligned}$$

We can conclude that $y = 1$, because otherwise the square root wouldn't exist. It follows that $x = \frac{3-1\pm 0}{2} = 1$, and then $z = 1$. □

Solution 2. Let's make variable substitution $u = x - 1$, $v = y - 1$, $w = z - 1$. We obtain the system

$$\begin{cases}(u+1)(v+1)(w+1) = uvw + 1\,, \\ (u-1)(v-1)(w-1) = uvw - 1\,,\end{cases}$$

(where the latter equation actually corresponds to the difference between two original equations).

After expanding all parentheses and reducing common terms we have

$$\begin{cases}uv + uw + vw + u + v + w = 0\,, \\ -(uv + uw + vw) + u + v + w = 0\,.\end{cases}$$

By taking the sum and the difference of these equations, we obtain $uv + uw + vu = 0$ and $u + v + w = 0$. Finally, observe that

$$u^2 + v^2 + w^2 = (u+v+w)^2 - 2(uv + uw + vw) = 0 - 0 = 0\,,$$

from which $u = v = w = 0$ follows, and $x = y = z = 1$. □

Solution 3. Consider the polynomial $f(t) = (t-x)(t-y)(t-z)$ with roots x, y, z. We can rewrite the system as

$$\begin{cases}-f(1) = -f(0) - 1\,, \\ -f(2) = -f(0) - 2\,.\end{cases}$$

Now consider the polynomial $g(t) = f(t) - f(0) - t$. Its main coefficient is 1, and 0, 1 and 2 are its roots. Hence $g(t) = t(t-1)(t-2)$. It follows that

$$\begin{aligned}f(t) &= g(t) + t + f(0) = t(t-1)(t-2) + t + f(0) = \\ &= t(t^2 - 3t + 3) + f(0) = t^3 - 3t^2 + 3t - 1 + f(0) + 1 = (t-1)^3 + f(0) + 1\,.\end{aligned}$$

Observe that $(t-1)^3+f(0)+1$ is an increasing function, which means that different real numbers cannot be its roots. So $x=y=z$ and also x is also the root of the derivative of $f(t)$. But $f'(t)=3(t-1)^2$, hence $x=y=z=1$. □

Problem 2. A convex quadrilateral $ABCD$ is circumscribed about a circle ω. Let PQ be the diameter of ω perpendicular to AC. Suppose lines BP and DQ intersect at point X, and lines BQ and DP intersect at point Y. Show that the points X and Y lie on the line AC. *(Géza Kós)*

Solution. The role of points P and Q is symmetrical, so without loss of generality we can assume that P lies inside triangle ACD and Q lies in triangle ABC.

Part 1. Denote the incircles of triangles of ABC and ACD by ω_1 and ω_2 and denote their points of tangency on the diagonal AC by X_1 and X_2, respectively. We will show that line BP passes through X_1, DQ passes through X_2 and $X_1=X_2$. Then it follows that $X=X_1=X_2$ is lying on AC (fig. 1).

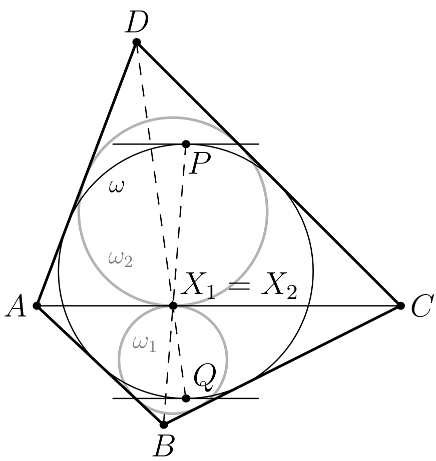

Figure 1: for the solution of the problem 2.

As is well-known, the tangent segments AX_1 and AX_2 to the incircles can be expressed in terms of the side lengths as

$$AX_1=\frac{1}{2}(AB+AC-BC) \quad \text{and} \quad AX_2=\frac{1}{2}(AC+AD-CD).$$

Since the quadrilateral $ABCD$ has an incircle, we have $AB+CD=BC+AD$ and therefore

$$AX_1-AX_2=\frac{1}{2}(AB-BC-AD+CD)=0;$$

this proves $X_1=X_2$.

By having the common tangents BA and BC, the circles ω are ω_1 are homothetic with center B. The tangents to ω at X_1 and to ω_1 at P are parallel, so this homothety maps P to X_1. Hence, the points B, P, X_1 are collinear.

Similarly, from the homothety that maps ω to ω_2, one can see that D, Q, X_2 are collinear.

Part 2. Now let γ_1 and γ_2 be the excircles of triangles of ABC and ACD, opposite to vertices B and D, respectively, and denote their points of tangency on the diagonal AC by Y_1 and Y_2, respectively. Analogously to the first part, we will show that line BQ passes through Y_1, DP passes through Y_2 and $Y_1 = Y_2$ (fig. 2).

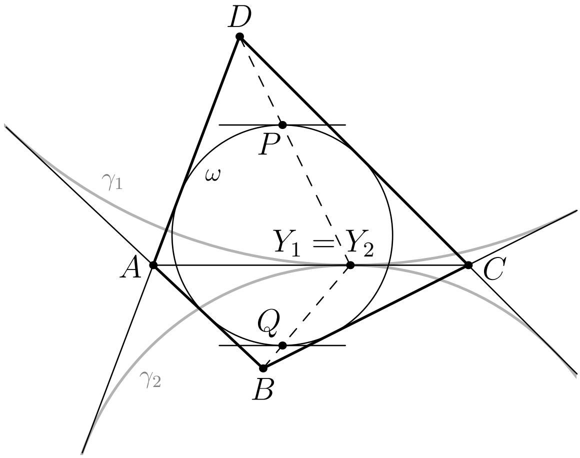

Figure 2: for the solution of the problem 2.

The tangent segments CY_1 and CY_2 to the excircles can be expressed as

$$CY_1 = \frac{1}{2}(AB + AC - BC) \quad \text{and} \quad CY_2 = \frac{1}{2}(AC + AD - CD);$$

by $AB + CD = BC + AD$ it follows that $CY_1 = CY_2$, so $Y_1 = Y_2$.

The circles ω and γ_1 are homothetic with center B. The tangents to ω and γ_1 at Q and Y_1 are parallel so this homothety maps Q to Y_1. Hence, the points B, Q, Y_1 are collinear.

Similarly, from the homothety that maps ω to γ_2, one can see that D, P, Y_2 are collinear. □

Problem 3. Let k be a positive integer such that $p = 8k + 5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1\leqslant i<j\leqslant 2k+1} (r_i^4+r_j^4)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

(Two integers are congruent modulo p if p divides their difference.) *(Fedor Petrov)*

Solution 1. We use the existence of a primitive root g modulo p, that is, such an integer number that the numbers $1, g, g^2, \ldots, g^{p-2}$ give all different non-zero remainders modulo p. Two powers of g, say g^m and g^k, are congruent modulo p if and only if m and k are congruent modulo $p-1$ (the "if" part follows from Fermat's little theorem and the "only if" part from g being primitive root).

There exist exactly $2k+1$ non-zero fourth powers modulo p, namely, $1, g^4, g^8, \ldots, g^{8k}$, thus the numbers $r_1^4, \ldots, r_{2k+1}^4$ are congruent modulo p to them in some order.

Define the map $f(j)\colon \{0,1,\ldots,2k\} \to \{0,1,\ldots,2k\}$ as a remainder of $2j$ modulo $2k+1$. Note that $8j$ and $4f(j)$ are congruent modulo $4(2k+1)=p-1$, therefore $g^{8j} \equiv g^{4f(j)} \pmod p$ for all $j=0,1,\ldots,2k$.

We have

$$\prod_{1\leqslant i<j\leqslant 2k+1} (r_j^4+r_i^4) = \prod_{1\leqslant i<j\leqslant 2k+1} \frac{r_j^8-r_i^8}{r_j^4-r_i^4} \equiv$$

$$\equiv \prod_{0\leqslant i<j\leqslant 2k} \frac{g^{8j}-g^{8i}}{g^{4j}-g^{4i}} \equiv \prod_{0\leqslant i<j\leqslant 2k} \frac{g^{4f(j)}-g^{4f(i)}}{g^{4j}-g^{4i}} \pmod p\,.$$

We may write $g^{4f(j)}-g^{4f(i)} = \pm(g^{4\max(f(j),f(i))}-g^{4\min(f(j),f(i))})$, where the sign is positive if $f(j)>f(i)$ and negative if $f(j)<f(i)$. Further, when the ordered pair (i,j) runs over all $k(2k+1)$ ordered pairs satisfying $0\leqslant i<j\leqslant 2k$, the ordered pair $\big(\min(f(j),f(i)),\max(f(j),f(i))\big)$ runs over the same set. Therefore the differences cancel out and the above product of the ratios $\prod \frac{g^{4f(j)}-g^{4f(i)}}{g^{4j}-g^{4i}}$ equals $(-1)^N$, where N is the number of pairs $i<j$ for which $f(i)>f(j)$. This in turn happens when $i=1,2,\ldots,k$; $j=k+1,\ldots,k+i$, totally $N=1+\ldots+k=k(k+1)/2$. Thus the result. □

Solution 2. Denote $t_i=r_i^4$. Notice that the set $T:=\{t_1,\ldots,t_{2k+1}\}$ consists of distinct roots of the polynomial $x^{2k+1}-1$ (over the field of residues modulo p). Let us re-enumerate T so that $t_{k+1}=1$, $t_i=1/t_{2k+2-i}$ for $i=1,2,\ldots,k$. The map $t\mapsto t^2$ is a bijection on T, the inverse map is $s\mapsto s^{k+1}$ and we naturally denote it $\sqrt{s}$. For distinct elements $t,s\in T$ we have $t+s=\sqrt{st}(\sqrt{s/t}+\sqrt{t/s})$. In the following formula $\prod$ denotes the product over all $k(2k+1)$ pairs of distinct elements

$t, s \in T$. We have

$$\prod(t+s) = \prod \sqrt{st} \cdot \prod\left(\sqrt{s/t} + \sqrt{t/s}\right) = \left(\prod_{t\in T} t\right)^k \cdot \left(\prod_{i=1}^{k}(t_i + 1/t_i)\right)^{2k+1}.$$

The first multiple equals 1 by Vieta's formulas for $x^{2k+1} - 1 = \prod_{t\in T}(x-t)$. As for the second multiple, note that there is a polynomial $\psi(x)$ with integer coefficients satisfying

$$\psi\left(x + \frac{1}{x}\right) = x^k + x^{k-1} + \ldots + 1 + \ldots + x^{-k}.$$

Obviously, the leading coefficient in ψ is 1. The constant term can be accessed by substituting the complex unit $x = \mathrm{i}$; the constant term is

$$\psi(0) = \psi\left(\mathrm{i} + \frac{1}{\mathrm{i}}\right) = \sum_{j=-k}^{k} \mathrm{i}^j = \begin{cases} 1 & \text{if } k \equiv 0,1 \pmod 4, \\ -1 & \text{if } k \equiv 2,3 \pmod 4. \end{cases}$$

The roots of ψ in the modulo p field are exactly $t_i + 1/t_i$, $i = 1, 2, \ldots, k$ (they are distinct). The product of the roots is

$$\prod_{i=1}^{k}(t_i + 1/t_i) = (-1)^k \cdot \psi(0) = \begin{cases} 1 & \text{if } k \equiv 0,3 \pmod 4, \\ -1 & \text{if } k \equiv 1,2 \pmod 4. \end{cases}$$

Finally, we conclude

$$\prod(t+s) = \begin{cases} 1 & \text{if } k \equiv 0,3 \pmod 4, \\ -1 & \text{if } k \equiv 1,2 \pmod 4. \end{cases}$$ □