The 4th Olympiad of Metropolises
Mathematics

Solutions

Day 1

Problem 1. Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p+n}{qr}, \frac{q+n}{rp}, \frac{r+n}{pq}$$

are integers. Prove that $p = q = r$. *(Nazar Agakhanov)*

Solution. We can assume without loss of generality that $p \geq q \geq r$.

Since $p \mid q+n$ and $p \mid r+n$, we have $p \mid q-r$, but $0 \leq q-r < q \leq p$, so we must have $q = r$. Furthermore, $q \mid p+n$ and $q \mid r+n$ imply $q \mid p-r = p-q$, so $q \mid p$, which is only possible if $p = q = r$. $\square$

Problem 2. In a social network with a fixed finite set of users, each user has a fixed set of *followers* among the other users. Each user has an initial positive integer rating (not necessarily the same for all users). Every midnight the rating of every user increases by the sum of the ratings that his followers had just before the midnight.

Let m be a positive integer. A hacker, who is not a user of the network, wants all the users to have ratings divisible by m. Every day, he can either choose a user and increase his rating by 1, or do nothing. Prove that the hacker can achieve his goal after some number of days. *(Vladislav Novikov)*

Solution. Let n be the number of users in the network. We will consider the remainders of ratings modulo m. The hacker wants to change all n remainders to zero.

If the hacker was allowed to increase ratings more than once on the same day then he could achieve his goal on the first day. Indeed, he could first increase the rating of the first user several times until it becomes 0 modulo m and then similarly for the second user, and so on. After that the hacker does not need any further action, the ratings of all users will be zeroes from the second day. We will call this *initial strategy.*

We will prove that the hacker can do the same operations but on different days, so that after some days the ratings of all users will be 0.

We represent the states of the network by vectors containing the modulo m remainders of the n users' ratings. Such vectors can be *added* modulo m. E.g., if $m = 4$

and $n = 6$, then

$$(1, 2, 3, 0, 2, 1) + (2, 3, 3, 0, 2, 3) = (3, 1, 2, 0, 0, 0) .$$

For a vector X of ratings before midnight we denote by $f(X)$ the new state after midnight.

Lemma.

$$f(X + Y) = f(X) + f(Y) .$$

Proof of the lemma. We need to show that vectors $f(X+Y)$ and $f(X)+f(Y)$ match at all the positions, i.e., for every person. Consider such a person and let his name be Bob. Let b_x be Bob's remainder in X, and let s_x be the sum of ratings of his followers modulo m in X. Let b_y and s_y be correspondingly Bob's remainder and the sum of remainders of his followers in Y.

Then in $f(X)$ Bob has $b_x + s_x$, in $f(Y)$ Bob has $b_y + s_y$, and in $f(X + Y)$ Bob has $b_x + b_y + s_x + s_y$. *Lemma is proven.*

Let us calculate the future effect of a hacker's action. If X is the initial vector and the hacker does not do anything, we obtain the sequence of vectors

$$X,\ f(X),\ f(f(X)),\ \ldots,\ f^k(X),\ \ldots$$

where f^k denotes the kth iteration of f.

If the hacker changes X by adding 1 to Bob's rating (and does nothing after that), we'll have

$$X + e_b,\ f(X + e_b) = f(X) + f(e_b),\ \ldots,\ f^k(X) + f^k(e_b),\ \ldots,$$

where 1 in the vector $e_b = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ corresponds to Bob.

So, the future effect after k days is $f^k(e_b)$. Similarly we show the following: let X be a vector, and suppose there are two strategies of the hacker that differ by addition of e_b on one of the days; then k days later the results of these strategies will differ by $f^k(e_b)$.

Each vector in the sequence $(f^k(e_b))$ is determined by the previous one and there is a finite number (namely m^n) of different vectors. Hence the sequence $(f^k(e_b))$ is eventually periodic. Let T_b be the length of its period. If we change the hacker's strategy by deferring the addition of e_b by T_b days to the future, then the result of the strategy will be the same after several days (no more than m^n days).

Now we will sequentially defer all actions from the first day to other days so that the hacker's actions are performed in distinct days. We can defer an event by the corresponding period as many times as we want, so it is possible.

Thus, we get a strategy for the hacker, the result of which after some time (no more than m^n days after the last change) does not differ from the initial strategy with all actions on the first day. The result of the initial strategy by this moment is the zero vector. So, it will be equal to zero in our new strategy with no more than one action per day. $\square$

Problem 3. In a non-equilateral triangle ABC point I is the incenter and point O is the circumcenter. A line s through I is perpendicular to IO. Line ℓ symmetric to the line BC with respect to s meets the segments AB and AC at points K and L, respectively (K and L are different from A). Prove that the circumcenter of triangle AKL lies on the line IO. *(Dušan Djukić)*

Solution 1. Let the incircle ω touch the lines BC, AC, AB and KL at points D, E, F and G, respectively. Denote the circumcenter of triangle AKL by U.

Since $DG \parallel IO$, we have $\angle KIG = \angle KFG = \angle FDG = \angle FDI + \angle IDG = \angle KBI + \angle OIG$ (in oriented angles) and hence $\angle KIO = \angle KBI$. It follows that the circle BIK is tangent to the line IO. Similarly, the circle CIL is tangent to the line IO.

Invert the diagram through ω (fig. 1). The points A, B, C, K and L map to the midpoints A', B', C', K' and L' of EF, FD, DE, FG and EG, respectively. The images of the circles BIK and CIL are the lines $B'K'$ and $C'L'$, so these two lines are parallel to IO. The circles ABC and AKL map to circles $A'B'C'$ and $A'K'L'$ whose centers are O_1 and U_1, respectively. Thus $O_1 \in IO$ and $U_1 \in IU$. Since $B'C'L'K'$

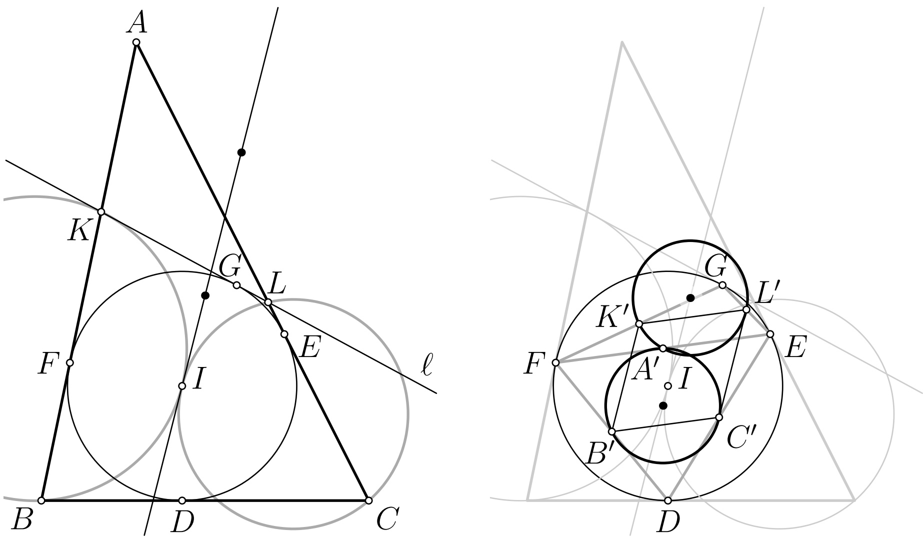

Figure 1: for the solution of the problem 3

is a parallelogram and $\angle L'A'K' = \angle FGE = 180° - \angle EDF = 180° - \angle B'A'C'$, the translation by vector $\overrightarrow{B'K'} = \overrightarrow{C'L'}$ takes the circle $A'B'C'$ to the circle $A'K'L'$, so the line O_1U_1 is parallel to IO. It follows that U_1 lies on IO, and so does U. □

Solution 2. Let the incircle ω touch the lines BC, AC, AB and KL at points D, E, F and G, respectively. Note that $GD \perp s$, so $GD \parallel IO$. Denote the circumcenter of $\triangle AKL$ by U. We use the following known fact:

Lemma. The line IO is the Euler line of triangle DEF.

Proof of the lemma. Let us apply the inversion with respect to the incircle ω. Points A, B and C map to the midpoints of EF, FD and DE, so the circumcircle of $\triangle ABC$ maps to the nine-point circle of $\triangle DEF$. Therefore the nine-point center of $\triangle DEF$ lies on IO, i.e. O lies on the Euler line of $\triangle DEF$. *Lemma is proven.*

By the Lemma, the centroid M of $\triangle DEF$ lies on IO. Similarly, we can show that the centroid N of $\triangle GEF$ lies on IU. However, $MN \parallel DG \parallel IO$, which implies that U lies on IO. □

Solution 3. We denote by $\delta(X, \overrightarrow{a})$ the oriented distance from point X to line $\overrightarrow{a}$.

Lemma. A point X in the plane lies on the line s if and only if

$$f(X) = \delta(X, \overrightarrow{BC}) + \delta(X, \overrightarrow{CA}) + \delta(X, \overrightarrow{AB}) = 3r\,,$$

where r is the inradius of $\triangle ABC$.

Proof of the lemma. The oriented distance to a fixed line is a linear function. Therefore $f(X)$ is a linear function as well (non-constant, since triangle ABC is not equilateral), so the locus of points X in the plane satisfying $f(X) = 3r$ is some line s'. Since $f(I) = 3r$, we have $I \in s'$.

Observe that, if ℓ_a, ℓ_b and ℓ_c are the perpendicular bisectors of BC, CA and AB directed towards the triangle, then $\delta(I, \ell_a) + \delta(I, \ell_b) + \delta(I, \ell_c) = \sum_{cyc}(p - c - \frac{a}{2}) = 0$. It follows by rotation by 90° that $f(X)$ remains constant as X traces any line perpendicular to IO (e.g. line s). *Lemma is proven.*

Denote the circumcenter of $\triangle AKL$ by U. Switching to $\triangle AKL$ with ω as the excircle opposite to A, we similarly deduce that:

Corollary. A point X in the plane lies on the line s_1 passing through I and orthogonal to IU if and only if

$$f_1(X) = -\delta(X, \overrightarrow{KL}) + \delta(X, \overrightarrow{LA}) + \delta(X, \overrightarrow{AK}) = 3r\,.$$

If $s \parallel BC$, triangle ABC is isosceles and the problem statement is trivial by symmetry. Now assume that the lines s, KL and BC meet at a point X_0. By the lemma, $f(X_0) = 3r$, hence $f_1(X_0) = 3r$, which in turn implies $X_0 \in s_1$. Therefore the lines s and s_1 coincide, i.e. U is on IO. □