The 4th Olympiad of Metropolises
Mathematics

Solutions

Day 2

Problem 4. There are 100 students taking an exam. The professor calls them one by one and asks each student a single question: "How many of 100 students will have a "passed" mark by the end of this exam?" The student's answer must be an integer. Upon receiving the answer, the professor immediately publicly announces the student's mark, which is either "passed" or "failed".

After all the students have got their marks, an inspector comes and checks if there is any student who gave the correct answer but got a "failed" mark. If at least one such student exists, then the professor is suspended and all the marks are replaced with "passed". Otherwise no changes are made.

Can the students come up with a strategy that guarantees a "passed" mark to each of them? *(Denis Afrizonov)*

Solution. The students can come up with the following strategy. Every student tells the number of students who already have the "passed" mark plus the number of students whose turn to answer has not come yet. In other words, if k students have already received a "failed" mark, the answer should be $99 - k$.

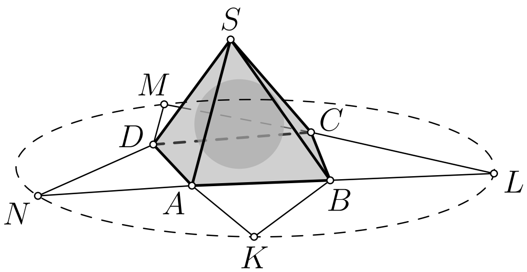
Let us prove that this strategy works. If all the students receive a "passed" mark, we are done. Otherwise consider the last (in order) student who has received a "failed" mark; denote him by Peter. Since all the students after him (if there are any) received a "passed" mark, then Peter's answer is correct. Thus the professor will get suspended and the students win.

Note. The students' winning strategy is unique.

To show this, consider the first student who deviated from our strategy; denote him by Basil. The professor can mark him "failed" and mark everyone after him "passed". All answers of the students before him will be rendered incorrect (according to the strategy, they will all give a larger answer), and his answer will be wrong, too (because the right answer would be given by the strategy). This means that Basil will remain "failed". □

Problem 5. We are given a convex four-sided pyramid with apex S and base face $ABCD$ such that the pyramid has an inscribed sphere (i.e., it contains a sphere which is tangent to each face). By making cuts along the edges SA, SB, SC, SD

and rotating the faces SAB, SBC, SCD and SDA outwards into the plane $ABCD$, we unfold the pyramid to the polygon $AKBLCMDN$ as shown in the figure. Prove that points K, L, M, N are concyclic.



(Tibor Bakos and Géza Kós)

Solution 1. Dilate the inscribed sphere from the point S in such a way that its image is tangent to the plane $ABCD$ from the opposite side; we obtain another tangent sphere exscribed to the base face of the pyramid (fig. 1). Denote the points

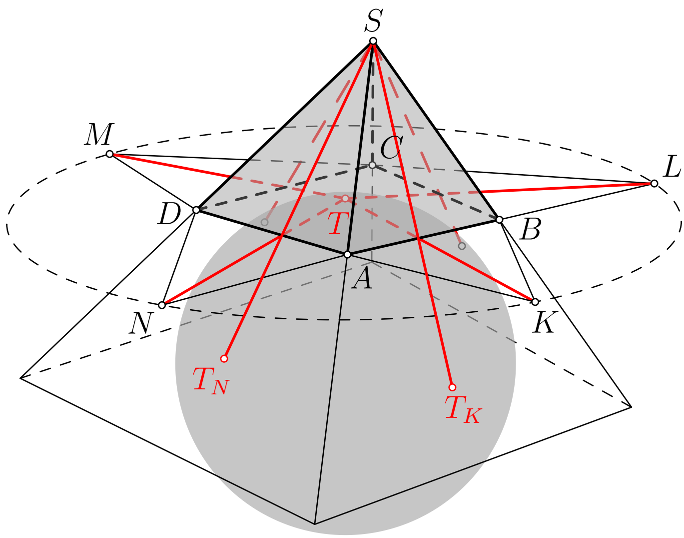


Figure 1: for the solution 1 of the problem 5

of tangency between the excribed sphere and the planes $ABCD$, ABS, BCS, CDS and DAS by T, T_K, T_L, T_M and T_N, respectively. We will show that $KT = LT = MT = NT$, so points K, L, M and N lie on a circle centred at T.

Notice that the quadrilaterals SBT_KA and $KBTA$ are symmetrical about the external dihedral angle bisector plane between faces $ABCD$ and ABS; this implies that $ST_K = KT$. We can see analogously that $ST_L = LT$, $ST_M = MT$ and $ST_N = NT$.

Moreover $ST_K = ST_L = ST_M = ST_N$ because these are tangent segments to the exsphere from S.

Hence, the segments ST_K, KT, ST_L, LT, ST_M, MT, ST_N, NT are all equal.

Note. The statement remains true for all circumscribed n-gon pyramids. For general case, we can repeat solution 1 or solution 2, without changes. Alternatively, one could reduce the general case to the case $n = 4$ by steps $n \mapsto n - 1$. □

Solution 2. We will use the same exsphere as in Solution 1. Denote its center by J. Point J lies in the external dihedral angle bisector plane between faces $ABCD$ and DAS, so $SJ = NJ$ (fig. 2). Repeating this observation for each side face, we can

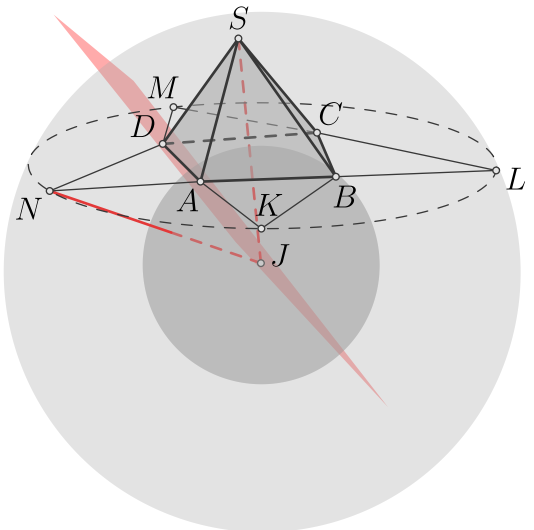


Figure 2: for the solution 2 of the problem 5

see that $SJ = KJ = LJ = MJ = NJ$.

Hence, the points S, K, L, M and N lie on a sphere with center J; therefore K, L, M and N lie in the common part of that sphere with plane $ABCD$ which is a circle. □

Solution 3. Denote the points of tangency between the inscribed sphere and the planes ABS, BCS, CDS and DAS by U_K, U_L, U_M and U_N, respectively.

The segment AS is rotated to AK and AN, so $AS = AK = AN$; analogously we have $BS = BK = BL$, $CS = CL = CM$ and $DS = DM = DN$ (fig. 3).

Notice that the triangles SAU_K and SAU_N are congruent, so $\angle ASU_K = \angle ASU_N$. Simlarly, we have $\angle BSU_L = \angle BSU_M$, $\angle CSU_M = \angle CSU_N$ and $\angle DSU_N = \angle DSU_K$.

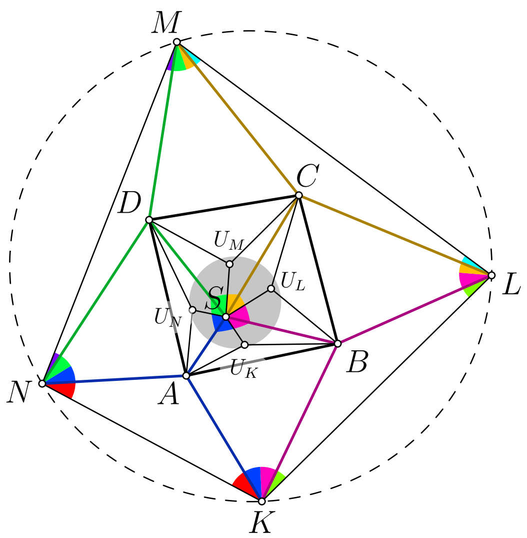


Figure 3: for the solution 3 of the problem 5

These together show that

$$\angle ASB + \angle CSD = \angle BSC + \angle DSA$$

so

$$\angle AKB + \angle CMD = \angle BLC + \angle DNA\,.$$

Together with the isosceles triangles ANK, BKL, CLM and DMN, we can see that in the quadrilateral $KLMN$ the sum of the opposite angles are equal, so the quadrilateral is cyclic. (Note that the the triangles ANK, BKL, CLM and DMN may degenerate or have opposite orientation. For a complete solution we need to use oriented angles or perform some case consideration.) □

Problem 6. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients. Assume that the numbers $f(1)$, $f(2)$, ..., $f(p)$ leave exactly k distinct remainders when divided by p, and $1 < k < p$. Prove that

$$\frac{p-1}{d} \leq k-1 \leq (p-1)\left(1-\frac{1}{d}\right).$$

(Dániel Domán, Gyula Károlyi and Emil Kiss)

Solution. For both inequalities we will use the standard

Fact 1. If $h(x)$ is a polynomial with integer coefficients and $m = \deg h < p-1$, then p divides $h(1) + \cdots + h(p)$.

Proof of the fact (sketch). We may find the integer coefficients c_0, c_1, …, c_m such that $h(x) = \sum_{i=0}^{m} c_i x(x-1)(x-2)\dots(x-i+1)$ (take the leading coefficient of h as c_m, then subtract $c_m x(x-1)\dots(x-m+1)$ from h and induct on degree.) Using the representation

$$x(x-1)\dots(x-s+1) = F_s(x+1) - F_s(x), \quad F_s(x) = \frac{x(x-1)\dots(x-s)}{s+1}$$

we get

$$\sum_{k=1}^{p} h(k) = \sum_{i=0}^{m} c_i(F_i(p+1) - F_i(1))$$

that is divisible by p, because $F_i(p+1) - F_i(1)$ is the difference of two fractions with the same denominators not divisible by p and with numerators equivalent modulo p. *This concludes the proof of the fact.*

(1) *Proof of* $\frac{p-1}{d} \le k-1$. Let u_1, …, u_k be all remainders given by $f(1)$, …, $f(p)$ modulo p. Denote $g(x) = (f(x) - u_1)\dots(f(x) - u_{k-1})$. The polynomial $g(x)$ takes exactly two values modulo p: 0 and $(u_k - u_1)\dots(u_k - u_{k-1})$. Then $\sum_{k=1}^{p} h(k)$ is not divisible by p and by Fact 1 we get $\deg g \ge p-1$, which is equivalent to $d(k-1) \ge p-1$.

To prove the rest of the statement, we will use the following well-known

Fact 2. Let $w_1, \dots, w_s$ be (not necessarily distinct) residues modulo p, and $s \le p-1$. Then the values modulo p of the power sums $r_j := w_1^j + \dots + w_s^j$ for $j = 1, \dots, s$ uniquely determine the multiset $\{w_1, \dots, w_s\}$.

Proof of the fact (sketch). Assume the contrary, i.e., that there exist two different multisets with the same remainders of $r_1, \dots, r_s$ modulo p. By removing the common elements from these multisets (and decreasing s accordingly), we reduce the fact to the case when our multisets $\{w_1, \dots, w_s\}$ and $\{u_1, \dots, u_s\}$ are disjoint.

The values of r_1, …, r_s modulo p uniquely determine the values modulo p of elementary symmetric polynomials

$$\sigma_0 = 1, \quad \sigma_k = \sum_{i_1 < \dots < i_k} w_{i_1} \dots w_{i_k} \quad \text{for } k = 1, \dots, s.$$

For example, this follows from Newton identities

$$k\sigma_k = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i} r_i$$

by induction in k (since $k \le s < p$, the remainder of $k\sigma_k$ modulo p uniquely determines the remainder of σ_k modulo p.) But the numbers $(-1)^i \sigma_i$, $i = 0, \dots, s$ are the coefficients of the polynomial $(x - w_1)\dots(x - w_s)$. On the other hand, the values of the polynomials $(x-w_1)\dots(x-w_s)$ and $(x-u_1)\dots(x-u_s)$ modulo p are distinct, for example, at $x=u_1$. Contradiction. *This concludes the proof of the fact.*

(2) *Proof that* $k-1 \le (p-1)(1-\frac{1}{d})$. Now denote $s := p-ks \le p-2$ and assume that $k-1 > (p-1)(1-\frac{1}{d})$ that reads as $ds < p-1$. The multiset $A = \{f(1), f(2), \dots, f(p)\}$ of residues modulo p may be represented as

$$A = (\{0, 1, \dots, p-1\} \setminus \{w_1, \dots, w_s\}) \cup \{u_1, \dots, u_s\},$$

where $w_1, \dots, w_s$ are the residues not taken by f, and $u_1, \dots, u_s$ are the residues taken more than once (with multiplicity taken into account). Since the polynomials $f(x), (f(x))^2, \dots, (f(x))^s$ have degrees less than $p-1$, using Fact 1 we get

$$\sum_{a\in A} a^j = \sum_{k=1}^{p} (f(k))^j \equiv 0 \pmod p, \quad j = 1, \dots, s.$$

On the other hand, modulo p we have

$$\sum_{a\in A} a^j = \sum_{i=1}^{p} i^j + \sum_{i=1}^{s} w_i^j - \sum_{i=1}^{s} u_i^j = \sum_{i=1}^{s} w_i^j - \sum_{i=1}^{s} u_i^j, \quad j \le p-2.$$

Therefore the multisets $\{w_1, \dots, w_s\}$ and $\{u_1, \dots, u_s\}$ coincide by Fact 2. Contradiction. □