# Problem 1: Newton-Laplace Cosmology

After Isaac Newton's discovery of the Law of Universal Gravitation, Pierre-Simón Laplace attempted to describe the behavior of the Universe as a cloud of moving matter. Laplace assumed that, in description of motion of this cloud, out of all interactions only gravity plays significant role at large distances between bodies. However, he lacked experimental data to accomplish construction of a realistic model.

Only in the 20th century, astronomical observations made it possible to establish that in the observed part of the Universe matter is distributed almost uniformly and isotropically (i.e. in the areas containing many galaxies the average density of matter is practically the same). In addition, Edwin Hubble discovered a law according to which distant objects move away from us along the line of sight with velocities $V_r$ proportional to distance $r$: $V_r = H \cdot r$, where $H$ is called the Hubble constant; it was found to be independent of distance and approximately equal to $H_t \approx 7 \cdot 10^{-11}$ years$^{-1}$. George Gamow suggested that such velocity distribution is due to the Big Bang, an explosion that occurred in a small area of space, and then the matter flew in different directions at different speeds. Therefore, the particles, which flew faster, have by now gone farther away from the explosion area.

1. Does the Hubble law imply that the solar system is in the area of the Universe where the Big Bang occurred? (Because Hubble made his observations from the Earth). Underline the correct answer and explain it with a drawing and formula.

In theoretical physics, Einstein's general relativity equations are used to describe the expansion of the Universe after the Big Bang. It is interesting, however, to find the conclusions Laplace would have drawn if he used Hubble law and information about the homogeneity of the Universe in his model based on Newton's laws. To find such conclusions, let us consider the expansion of the so-called Newtonian Universe (NU). The NU is a homogeneous ball of total mass $M = 10^{55}$ kg, in which particles of matter interact due to the Newton's law of gravity with gravitational constant $G \approx 6{,}7 \cdot 10^{-11}$ m$^3$/(kg·s$^2$), and velocities of matter are distributed according to Hubble law, in which the "constant" $H$ is actually a function of time $H = H(t)$.

2. Calculate gravitational potential energy $E_g$ of the NU at a time $t$ when its radius is equal to $R$ (give your answer as a formula expressing $E_g$ in terms of $M$ and $R$).
3. Calculate the kinetic energy $E_K$ of the NU at the same time (give your answer as a formula expressing $E_K$ in terms of $M$, $H$ and $R$).

It is clear that in the process of further expansion (over time $t$) the matter in the NU will be decelerated by gravity. Let's assume that at time $t$, which is counted from the Big Bang, some "residents" of the NU measured the average density $\rho(t)$ of matter in it and the Hubble constant $H(t)$.

4. At what relationship between $\rho(t)$ and $H(t)$ the expansion of the NU will stop and be replaced by compression in finite time? Give the answer in the form of inequality.
5. Let the total energy of matter in the NU, i.e. the sum of kinetic energy and potential energy of gravitational interaction, be equal to $E = -\frac{2}{15} M c^2$, where $c$ is the speed of light in vacuum. Find the maximum radius of the NU in the process of expansion. Write down the formula and get a numerical answer in parsecs (1 parsec is approximately equal to 3.2 light years, or $3 \cdot 10^{16}$ m).
6. For the conditions described in question 5, find the total lifetime of the NU from the Big Bang to the Big Implosion. Write down the formula and get a numerical answer in years.

**Mathematical hint**: $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}} dx = (y = \sqrt{1-x}) = 2\int_0^1 \sqrt{1-y^2} dy$.

**7.** What are the options for further movement of matter in the ball at different ratios between the density and the Hubble constant? Describe the qualitative behavior of the NU radius for each of the possible cases. To do this, depict in Figures 1a-1b, 2a-2b and 3a-3b three different pairs of graphs showing the NU radius $R$ and the rate of its expansion $V_R = \frac{dR}{dt}$ versus time (keep in mind that at the time of the Big Bang $t = 0$ and the radius of the NU is considered to be almost zero). Draw the graphs qualitatively, showing all the important details and features.

Let the results of observations imply that in terms of energy the NU under study has arisen "from nothing", that is, the total energy of matter in it is equal to zero.

**8.** Find the expansion law $R(t)$ of such NU. Give the formula in your answer.

**9.** How will the density of matter and the Hubble constant change in time? Write down formulas for $\rho(t)$ and $H(t)$.

Suppose the NU with zero energy expands adiabatically and reversibly. Also, let's assume that its entropy $S$ is related to its volume $V$ and temperature $T$ according to the ideal gas formula, i.e. $S(V,T) \approx const \cdot M \cdot \ln\left(\frac{VT^{3/2}}{V_0 T_0^{3/2}}\right)$. At present time, when the Hubble constant in the NU is equal to its modern value of $H_t \approx 7 \cdot 10^{-11}$ years$^{-1}$, its temperature is equal to $T_t \approx 2{,}7$K.

**10.** Find the NU temperature at time $t_0 = 1$ s after the Big Bang. Give a numerical answer.