Problem 3: «Monopoles and Dyons»

As early as in the 18th century Charles Coulomb established experimentally that the laws of interaction of electric charges and poles of a magnet are identical. After that many researches tried to separate two poles of magnet to obtain a «single» *magnetic charge* which was named «*magnetic monopole*». However, any known magnet ranging from a macroscopic size to elementary particle turns out to be *magnetic dipole*, i.e. it always has the both magnetic poles. In the 19th century Andre-Marie Ampere showed that the field of magnetic dipole was generated by a small loop of electric current, therefore, it was not necessary to assume existence of magnetic charges in order to describe the observed phenomena. In the 20th century Paul Dirac proposed an idea of how to make an «artificial» magnetic monopole: one should take very thin and very long (of a length l and a cross-section S, so that $l \gg \sqrt{S}$) flexible solenoid, which winding consists of N turns and carries a steady current I. The magnetic flux flowing inside the solenoid spreads out of its end radially in all directions. Therefore, the magnetic field near the solenoid end (everywhere except on the solenoid itself) is identical with the field of a magnetic monopole of magnetic charge $M = \frac{NIS}{l}$. The magnetic induction of the monopole equals $\vec{B} \approx \frac{\mu_0 M}{4\pi r^3}\vec{r}$, where $\mu_0 \approx 4\pi \cdot 10^{-7} H/m$ is the magnetic constant. Later this idea was implemented experimentally: with a vortex in a quantum liquid or gas serving as a solenoid. Two «ends» of such a solenoid are far away and move almost independently, thereby creating «illusion» of two monopoles resided in the system. Lines of magnetic induction go from the solenoid ends radially and spread spherically symmetrically as the lines of electric field of a point-like electric charge.

In 1974 Alexander Polyakov and Gerard t'Hooft discovered that «real» magnetic monopoles with a nonzero magnetic charge must exist in some realistic theories of elementary particles. Therefore, the active experimental search for such particles continues even in the 21st century.

Part I: Detection of Passage of Magnetic Monopole.

One of several methods of magnetic monopole detection is to register an electric current induced by monopole passing through an conductive loop. Currently the monopole detectors use superconductor loops (aka SQUID – Superconducting Quantum Interference Device).

1.1. A magnetic monopole travels at a large speed along the axis of a thin *superconducting* ring and passes through the ring. Plot the dependence of an induction current I on time t in Figure 1. Let the positive direction of the current correspond to monopole approaching the ring. Assume that at $t = 0$ the monopole is at the ring plane. Your sketch must explicitly show all important features.

Let a magnetic charge of the monopole be $M = (4 \cdot 10^{-8}\, C \cdot H \cdot s^{-1})/\mu_0$ and its mass be $m = 1$ μg, the ring radius be $a = 1$ m, and assume the ring to be made of a wire of radius $r = 0{,}5$ mm. The ring inductance is well approximated by Grover's formula: $L \approx \mu_0 a \left[\ln\left(\frac{8a}{r}\right) - \frac{7}{4}\right]$.

1.2. Assume that monopole velocity at a large distance ($x_0 \gg a$) from the ring is $v_0 = 1$ km/s. Let x be a current distance between the monopole and the ring. The whole setup is in vacuum and the monopole travels along the ring axis. Determine the induction current $I(x)$ at this moment. What is the maximum value I^S_{max} of the induction current? Neglect electromagnetic radiation emitted by the system and determine the maximum variation of monopole velocity ($v_0 - v_{min}$) during the motion. Your answer must include an equation for I, an equation and the numerical value for I^S_{max} (in mA), and an equation and the numerical value for the velocity variation (in m/s).

Mathematical hint: The solid angle subtended by a cone with the apex angle 2α equals $\Omega = 2\pi \cdot [1 - \cos\alpha]$.

Then you should study how electrical resistivity of the ring would affect a shape of the pulse of induction current and the monopole motion.

1.3. Assume that the magnetic monopole travels at a large velocity along the axis of a thin *conducting* ring and passes through the ring.

A) Plot the dependence of induction current *I* through the ring on time *t* in Figure 2. An electrical resistance of the ring is large: $R \gg \frac{Lv_0}{2r}$.

B) Plot the dependence *I*(*t*) for the same monopole motion and a «medium» ring resistivity, $\frac{Lv_0}{a} \ll R \ll \frac{Lv_0}{2r}$, in Figure 3.

C) For comparison, plot in Figure 4 the dependence of *I*(*t*) for the case of a thin linear cylindrical magnet passing through the ring with a large electrical resistance (the magnetic axis of the magnet is aligned with the ring axis).

In all cases the positive direction of induction current corresponds to an object approaching the ring. Assume that a monopole (or the center of the cylindrical magnet) is at the ring plane at $t = 0$. Draw qualitative graphs which exhibit all important features of a plotted dependence.

Consider a situation described in **1.2**, but for a ring with an electrical resistivity $\rho = 10^{-8}\ \Omega \cdot m$ (and the same the numerical values of $a, r, M, m,$ and v_0).

1.4. At some moment the monopole is at a distance $x \gg r$ from the ring center and travels at a speed $v(x)$. What is the value of induction current *I*(*x*) at this moment? Neglect emission of electromagnetic waves and estimate the change of monopole velocity after the monopole has passed through the ring (i.e. when the monopole is already far away from the ring). Your answer must include the equation for *I* and the numerical value of the velocity change in m/s.

Part II: Dyon and Circular Orbits.

In addition to monopoles, many theories of high energy physics predict hypothetic particles carrying both electric and magnetic charges (aka *dyons*).

Suppose that a light charged particle of a mass m and an electric charge q travels in the field of a very heavy dyon with an electric charge $Q > 0$ and a magnetic charge $M > 0$. Recall that the vacuum permittivity ε_0 in Coulomb's law is related to the magnetic constant as $\frac{1}{\varepsilon_0 \mu_0} = c^2$, where $c \approx 3 \cdot 10^8$ m/s is the velocity of light in vacuum.

2.1. The particle is set into motion so that it travels in the field of the dyon in a circular orbit at a constant velocity $v \ll c$ (emission of electromagnetic waves at such a velocity is negligible). Determine a possible radius of the orbit. The answer must be represented by a formula. Sketch the dyon, the particle, and its orbit.

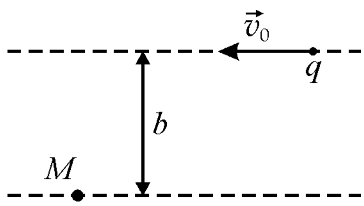
2.2. Show the orbit location relative to the dyon (determine all necessary geometrical parameters and write down the corresponding formulae). Indicate these parameters on the plot sketched in **2.1**.

2.3. Determine two integrals of motion (a scalar and a vector) which remain constant during the motion of electric particle in the dyon field. Express these integrals via a particle coordinate $\vec{r}$ and a velocity $\vec{v}$ (these formulae must also include the parameters of the system). Derive the formulae for the integrals of motion for the circular orbit studied above (the answer must be given as an equation which includes the system parameters and v).

Hint. The vector integral at $M = 0$ must become angular momentum of the particle which is obviously conserved during the particle motion in the field of a point-like electric charge. When $M \neq 0$ the vector integral equals the net angular momentum of the system consisting of the dyon, the particle, and their electromagnetic field.

Mathematical hint. The following identity can be helpful: $\frac{d}{dt}\left(\frac{1}{r}\right) = -\frac{\left(\frac{dr}{dt}\right)}{r^2} = -\frac{(\vec{v}\vec{r})}{r^3}$.

Part III: Magnetic Monopole and Quantization of Electric Charges.



Consider a particle of mass *m* and electric charge *q* traveling in the field of an immobile monopole with a magnetic charge *M*. Let the particle start its motion far away from the monopole at a velocity v_0, and an *impact parameter b* (this is the perpendicular distance between the monopole and the particle extrapolated path, see the Figure). It turns out that the value of *b* was such that after the particle has passed by the monopole and moved far away from it, it travels in a plane parallel to the plane of the Figure at a distance *d*.

3.1. Determine a particle velocity v' after it was scattered on the monopole (i.e. when the particle was far away from the monopole). The answer must be given as a formula including the quantities introduced above.

3.2. Determine all possible values of the scattering angle (the angle between $\vec{v}'$ and $\vec{v}_0$). The answer must be given as a formula including the quantities introduced above.

3.3. Determine the perpendicular distance b' between the monopole and the particle path after the scattering. The answer must be given as a formula including the quantities introduced above.

In classical physics magnetic and electric charges can take arbitrary values, however, in *quantum theory* this is no longer true. The point is that in quantum theory the angular momentum of a particle (aka *orbital angular momentum*) and the angular momentum of electromagnetic field obey the rule of quantization of angular momentum: any vector component of angular momentum must be a multiple integer of $\frac{\hbar}{2} \equiv \frac{h}{4\pi}$, where $h \approx 6{,}63 \cdot 10^{-34}$ J·s is the fundamental Planck constant.

3.4. Consider a particle with an electric charge *q* moving in the field of magnetic charge *M* and use the law of quantization of angular momentum. Determine the condition relating *q* and *M*, such that the conservation laws specific for this problem and the law of quantization of angular momentum do not disagree. Determine a value of magnetic charge for which any electric charge is a multiple of the elementary charge $e \approx 1{,}6 \cdot 10^{-19}$ C. The answer must be a formula which relates magnetic and electric charges and the numerical value of *M* in C·m/s.