Reference materials and practice exercises
for problem 3 «QUBITS AND PHOTONS»

Dear contesters! Please, read carefully this material and do the exercises under the guidance of your team supervisors. This will help you to solve problem 3 of the Olympiad.

You will be allowed to use these materials during the theoretical tour of the Olympiad!

PROBABILITY AND STATISTICS

When we study a complicated phenomenon, the outcome of a specified event often cannot be exactly predicted. However, when the number of such events is large we can employ *statistical approach*. In this case we use the concept of *probability* of event. To do this we specify a set of some «elementary» events which are usually considered as *equiprobable*. Then we calculate the fraction of the total number of «elementary» events corresponding to happening of the specified event. This fraction is referred to as a probability of the event.

For instance, consider a throw of a uniform cubic dice with its faces numbered from 1 to 6. It is very difficult to calculate the outcome of a single throw (i.e. to say which number will be on top when the dice stops). However one could notice that a single throw can result in six possible outcomes (i.e. each of six outcomes is regarded as an «elementary» event) and if they are equiprobable, each number appears with a probability $w = \frac{1}{6}$. Multiplying the probability by the number of trials N we obtain an *expectation value* of the number of events happened, wN (or an average number of «successful» outcomes in a series of N trials). Of course, this does not mean that for $N = 6$ of dice throws number 1 comes out once. This means that given a very large number N of throws, the number of 1 outcome approaches $\frac{1}{6}N$.

Mathematicians have proven that in a series of N independent trials with a probability of success equal to w, the *standard deviation* of the number of successes (i.e. square root of the mean square of a deviation from the expectation value) equals $\Delta N = \sqrt{Nw(1 - w)}$. It follows that the number of outcome 1 for 6 throws is $N_1 \approx 1,0 \pm 0,9$ (the relative error is close to 90%!) and for 60000 throws it is $N_1 \approx 10000 \pm 91$ (the relative error is less than 1%).

Suppose the dice has been thrown twice. How can one determine the probability that the number 1 comes out at the first throw and the number 2 at the second? This probability is calculated from the assumption that this particular outcome is one of 36 possible outcomes of two independent throws: we see that probability for two independent events happening together is the product of the probabilities of each of the events.

Exercise 1: Calculate the probability that number 1 comes out more than once in a series of 6 throws.

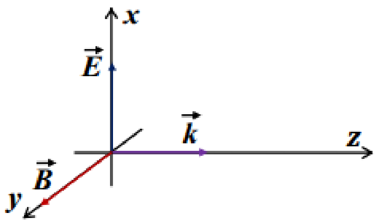
BITS AND QUBITS

Bit is not only a unit of information. From physics perspective, bit is also a common name of any *classical* system with two states. For instance, of any circuit with a power source and a switch. When the switch is closed, there is a current flowing in the circuit (state 1), when opened there is no current (state 0). Taking a set of such circuits with opened and closed switches it is possible to «record» any binary code as a sequence of zeros and units.

Qubit (*quantum bit*) is a *quantum* system with two basic states which we denote as |0 and |1. A fundamental difference between qubit and bit is due to the nature of a quantum object which can exist in a state of *superposition*, i.e. a linear combination (mixture) of the basic states. The second important difference is that result of a *measurement* is only *statistically* related to a state of quantum system; a measurement of «current» in a circuit (bit) in the state 1 always detects the current, and the measurement in the state 0 always detects no current, another outcome is impossible. A measurement done on a qubit, which state is a superposition of basic states |0 and |1, will produce the result corresponding to state |0 with a probability w_0 and the result corresponding to |1 with a probability w_1. Since there are no other basic states, one of this outcomes must be realized, so $w_0 + w_1 = 1$.

PHOTONS AS QUBITS

According to quantum theory, light can be considered as a bunch of discrete «quanta» of electromagnetic field, *photons*. To specify a state of single photon, one should specify its frequency ν, a direction of motion (e.g. by a unit vector $\vec{n}$), and a *polarization state*. The frequency and the direction of motion can be specified together by a *wavevector* $\vec{k} \equiv \frac{2\pi}{\lambda}\vec{n} = \frac{2\pi}{c}\nu\vec{n}$ (here λ is a wavelength and c is a speed of light). Wavevector determines a momentum $\vec{p} = \hbar\vec{k}$ and energy $\mathcal{E} = h\nu = \hbar c|\vec{k}|$ of photon (here $\hbar \equiv \frac{h}{2\pi} \approx 1,05 \cdot 10^{-34}$ J·s is Planck's constant). For any $\vec{k}$, a classical electromagnetic wave has two independent polarization states. They correspond to two possible orthogonal directions of electric field vector $\vec{E}$ of the wave (in a plane orthogonal to $\vec{k}$). For instance, if $\vec{k}$ points along z, the wave can have two independent (basic) polarization states along x and y.



Of course, another polarization states are also possible but they can be constructed as *superpositions* of basic states. Note that superposition allows one to construct not only other *linear* polarizations (for those vector $\vec{E}$ is directed along an axis different from both x and y and the components of $\vec{E}$ are not equal but oscillate *in phase*). Superposition also allows one to construct *circular* or *elliptic* polarizations for which the components of $\vec{E}$ oscillate with a relative phase shift.

It turns out a single photon also has polarization states. For a photon with a wavevector $\vec{k} = k \cdot \vec{e}_z$ there are also two independent polarization states: $|x$ and $|y$. The corresponding classical harmonic electromagnetic wave polarized along x consists of many photons in the state $|x$. Thus if a photon wavevector is known (we direct z-axis along $\vec{k}$) and only polarization related quantities are measured, a photon can be considered as a qubit with the basic states $|x$ and $|y$. In addition, a photon can exist in a state which a superposition of the basic states. In such a state a measurement of photon polarization must detect x-polarization with a probability w_x and y-polarization with a probability $w_y = 1 - w_x$.

COMPLEX NUMBERS

Setting the goal of calculating square root of a negative number, one will define the imaginary unit, $i \equiv \sqrt{-1}$, and then the whole set of complex numbers $z = x + iy$. In this notation x and y are two real numbers known as real and imaginary parts of a complex number z: $x = Re(z)$, $y = Im(z)$. Complex numbers can be multiplied and divided, a result will be another complex number. The corresponding operations are defined by the following formulae:

$$z_1 \bullet z_2 = \left(x_1 + iy_1\right)\left(x_2 + iy_2\right) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{\left(x_2+iy_2\right)(x_2-iy_2)} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

In general, any algebraic operation on complex numbers is allowed. Many common functions can be defined as functions of complex variables. For instance, exponential function satisfies to *Euler's formula*:

$$e^z = e^x \bullet e^{iy} = e^x \bullet [\cos cos\,(y) \ + i \bullet sin(y)].$$

A complex number $z^* \equiv x - iy$ is called a conjugate of z. A real non-negative quantity $|z| \equiv \sqrt{z \bullet z^*} = \sqrt{x^2 + y^2}$ is called a *magnitude* of the complex number. Introducing ana *argument* φ of a complex number as

$$\left\{cos(\varphi)\equiv\frac{x}{\sqrt{x^2+y^2}}\ sin(\varphi)\equiv\frac{y}{\sqrt{x^2+y^2}}\ \right\}$$

it is possible to write a complex number in exponential form: $z = |z|\cdot e^{i\varphi}$.

Exercise 2: Write down complex number $4 + 3i$ in exponential form.

Exercise 3: Calculate $\sqrt{1 + i\sqrt{3}}$.

MATRICES

An $(n\times m)$-matrix is a rectangular array with n rows and m columns. For instance, a (2×1)-matrix is a «two-component» column $U = \left(u_1\ u_2\right)$, a (1×2)-matrix is a «two-component» row $V = \left(v_1\ v_2\right)$, and a (2×2)-matrix is a square matrix

$\hat{A} = \left(a_{11}\ a_{12}\ a_{21}\ a_{22}\right)$. Let us learn how to multiply matrices. This is done according to the «row by column» rule: the product of an $(n\times m)$-matrix $\hat{A}$ and an $(m \times l)$-matrix $\hat{B}$ equals an $(n\times l)$-matrix $\hat{C}$ which entries are $c_{ij} = \sum_{k=1}^{m} a_{ik} \bullet b_{kj}$. For example,

$$(1\ 2\ 3\ 4\ ) \bullet (5\ 6\ ) = (1\cdot5\ +\ 2\cdot6\ 3\cdot5\ +\ 4\cdot6\ ) = (17\ 39\ ).$$

Note that two matrices can be multiplied only if the number of columns of the first equals the number of rows of the second! A matrix $\hat{A}^\dagger$ obtained by transposing rows and columns of matrix $\hat{A}$ and by replacing its entries with their complex conjugates is called conjugate to $\hat{A}$: $\hat{A}^\dagger \equiv \left(a_{11}^*\ a_{21}^*\ a_{12}^*\ a_{22}^*\right)$. For example, $(1\ 2i\ 3\ -\ 4i\ )^\dagger = (1\ 3\ -\ 2i\ 4i\ )$, and $(0,6\ 0,8i\ )^\dagger = (0,6\ -\ 0,8i\ )$.

Exercise 4: Calculate $(1\ 2i\ 3i\ 4\ ) \bullet (4\ -\ i\ -\ 3i\ 2\ )$.

Exercise 5: Calculate $(3i\ 3\ )^\dagger \bullet (1\ -\ i\ i\ 2\ ) \bullet (2i\ 2\ )$.

PHOTON POLARIZATION

Now we are in a position to describe polarization states of photon as a qubit. Let a photon state $|x$ correspond to a two-component column $(1\ 0\ )$, and a state $|y$ to a two-component column $(0\ 1\ )$. Then an arbitrary superposition of these states is represented by a column $\alpha(1\ 0\ ) + \beta(0\ 1\ ) = (\alpha\ \beta\ )$ in which complex numbers α and β are related by *normalization condition,* $|\alpha|^2 + |\beta|^2 = 1$. This condition is physical since a quantity $|\alpha|^2 \equiv w_x$ is interpreted as a probability to detect photon polarization along x-axis and a quantity $|\beta|^2 \equiv w_y$ as a probability to detect photon polarization along

y-axis. Note that the numbers α and β must be complex because two real numbers are not enough to describe correctly physical phenomena which include photons.

The set of all such columns can be regarded as a two-dimensional vector space (surely, an unusual one: do not forget that «coordinates» α and β in this space are complex numbers!). A *dot-product* of two vectors in this space can be defined as $\langle 2\rangle \equiv \left(\alpha_1\, \beta_1\right)^\dagger \bullet \left(\alpha_2\, \beta_2\right) = \left(\alpha_1^*\, \beta_1^*\right) \bullet \left(\alpha_2\, \beta_2\right) = \alpha_1^*\alpha_2 + \beta_1^*\beta_2$. It is worth noting that: 1) basic states are orthogonal (their dot-product is zero, $\langle y\rangle = 0$); 2) normalization condition can be interpreted as a requirement that the square of «vector» magnitude equals unity; 3) a probability can be written as a squared magnitude of dot-product, e.g. $w_x = |(1\ 0\ ) \bullet (\alpha\ \beta\ )|^2$.

States of qubit described by two-component spinors are called «*pure*» states. In the most general case, a probability to detect a qubit in pure state |2, while the qubit is in pure state |1, equals $w_{21} = |\langle 1\rangle|^2$. It is important to note that a probability evaluated for a pure state according to this formula do not change if the corresponding column is multiplied by a complex number $e^{i\varepsilon}$ (with any real ε). Besides, the normalization condition remains valid (both statements follow from $|e^{i\varepsilon}|^2 = cos^2(\varepsilon) + sin^2(\varepsilon) = 1$). Therefore, it is impossible to determine experimentally an «overall phase» of a pure single-photon state. And this means that the overall phase can be arbitrarily assigned, e.g. α can be chosen to be real and positive, then a column specifying a general pure state is $\left(\alpha\ |\beta|\ \bullet e^{i\varphi}\right)$.

Exercise 6: A photon is in a pure state $\left(1/\sqrt{2}\ (1\ -\ i)/2\ \right)$. Determine a probability to detect it in a pure state $\left(2\sqrt{2}/3\ i/3\ \right)$.

Now it's time to remember that in classical physics an electromagnetic wave can be both polarized and unpolarized. The latter is considered as a mixture of incoherent waves with all polarizations possible. Can a single «unpolarized photon» exist? Experiment shows that a photon can be in a state which does not have any polarization. Measurement of any quantity related to polarization in such a state yields a value corresponding to **any** linear polarization (along **any** axis orthogonal to k) with equal probability.

Wave analogy here is not accidental: any quantum object has wave properties associated with a *wave of probability* to find the object in some state. These waves can be incoherent as well and only coherent waves can participate in a quantum superposition. When *quantum coherence* is lost, the corresponding polarization states of photon cannot be described by any column (α β) because the latter corresponds to a superposition of coherent probability waves. To specify an incoherent mixture, another object is used, a (2×2)-matrix:

$$\hat{\rho} = \left(\rho_{11}\, \rho_{12}\, \rho_{21}\, \rho_{22}\right).$$

Entries of this matrix obey the following relations: $\rho_{11} + \rho_{22} = 1$ and $\rho_{11} \bullet \rho_{22} > \rho_{12} \bullet \rho_{21}$; here ρ_{11} and ρ_{22} are real and positive and ρ_{12} are ρ_{21} complex conjugates ($\rho_{21} = \rho_{12}^*$). These properties follow from physical requirements: in particular, $\rho_{11} = w_x$ and $\rho_{22} = w_y$, so the probabilities are positive and their sum equals 1. A completely unpolarized state of photon is given by $\hat{\rho}_{un} = (1/2\ 0\ 0\ 1/2\ )$.

A qubit state which cannot be described as a two-component column (and requires a 2×2-matrix) is called «mixed». In general, the probability that a cubit in a mixed state $\hat{\rho}$ will be detected in a pure state $(\alpha\ \beta)$ equals $w = \left(\alpha^{*}\ \beta^{*}\right)\hat{\rho}(\alpha\ \beta)$.

Exercise 7: Which of these 2×2-matrices corresponds to a mixed state of photon polarization?

1) $(3/5\ \ -i/2\ i/2\ 2/5)$; 2) $(1/4\ i/4\ \ -i/4\ 3/4)$; 3) $(1/3\ \ -1/5\ \ -1/5\ 2/5)$;

4) $(1/3\ \ -i/3\ \ -i/3\ 2/3)$; 5) $(-1/3\ \ -i/3\ i/3\ 4/3)$.

Exercise 8: A photon is in a mixed state $(1/3\ \ -i/3\ i/3\ 2/3)$. Determine the probability to detect it in a pure state $\left(2\sqrt{2}/3\ i/3\right)$.

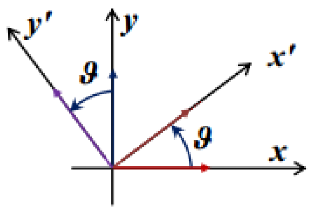
EVOLUTION AND MEASUREMENTS

Let us discuss what we can do with a state of photon polarization. Actually there are only two possibilities:

(1) we can create conditions in which a photon polarization state will change in a certain way (*evolve*);

(2) we can measure a physical quantity related to polarization.

For instance, we can direct a beam of photons in a medium which rotates a plane of photon polarization by an angle proportional to the length of traversed path. In this case, how will a polarization state change? If the rotation angle $\vartheta(z) = kz$, the state $|x = (1\ 0)$ must transform into a state polarized along *x*' (see the Figure) and the state $|y = (0\ 1)$ into a state polarized along *y*' (as usual, the positive direction of rotation is counterclockwise).



It is not difficult to understand that the «new» states can be written as linear combinations of the «old» ones:

$$(1\ 0) = |x \rightarrow |x' = cos(\vartheta)|x + sin(\vartheta)|y = (\cos cos\,(\vartheta)\ \ \sin sin\,(\vartheta)\ ),$$

$$(0\ 1) = |y \rightarrow |y' = cos(\vartheta)|y - sin(\vartheta)|x = (-\sin sin\,(\vartheta)\ \ \cos cos\,(\vartheta)\ ).$$

Clearly, this transition can be described as the result of multiplication of a «matrix of rotation of polarization plane»,

$\hat{U}(\vartheta) = (cos(\vartheta)\ \ -sin(\vartheta)\ sin(\vartheta)\ cos(\vartheta))$, by a column corresponding to the initial state. Thus, $(cos(\vartheta)\ sin(\vartheta)) = \hat{U}(\vartheta)(1\ 0)$ and $(-sin(\vartheta)\ cos(\vartheta)) = \hat{U}(\vartheta)(0\ 1)$.

Therefore, evolution of any superposition of basic states of a wave passing through a medium, which rotates plane of polarization, is described as action of $\hat{U}(\vartheta)$ on the initial (column) state. In the same way, any evolution of a polarization state of photon is described by the corresponding *evolution matrix* $\hat{U}$ which action specifies this evolution.

Exercise 9: A photon was in a state $(3/5\ 4i/5)$. It passes a layer of material rotating polarization plane counterclockwise by 30°. What is the probability to detect the photon in a state polarized along y-axis?

In a measurement process photon interacts with detector and this interaction changes a photon state: the detector «forces» the photon into a state with a certain value of the quantity being measured (before the measurement the photon could to have not a certain value of this quantity). Mathematical description of measurement of a quantity related to photon polarization also involves 2×2-matrices: every detector is described by the corresponding *matrix of «observable»* $\hat{F} = \left(f_{11}\, f_{12}\, f_{21}\, f_{22}\right)$, such that allowed values (no more than two because there are only two independent polarization states) of the measured quantity F are determined from the equation:

$$F^2 - (f_{11} + f_{22})F + f_{11}f_{22} - f_{12}f_{21} = 0. \qquad (1)$$

The set of allowed values of an «observable» is called its *spectrum*. Note that the allowed values of a physical observable must be real, therefore diagonal matrix elements of the «observable» (f_{11} and f_{22}) must be real and non-diagonal must be complex conjugates ($f_{21} = f^*_{12}$). A state in which the «observable» has a certain value corresponds to a normalized column which satisfies an equation

$$\hat{F}(\alpha\ \beta) = F(\alpha\ \beta). \qquad (2)$$

(actually, it is not difficult to verify that equation (1) is the condition for equation (2) to have non-trivial solutions). By means of equations (1) and (2), for a given matrix of an «observable», one can determine the allowed values $F_{1,2}$ of the observable and the states $|F_1 = \left(\alpha_1\ \beta_1\right)$ and $|F_2 = \left(\alpha_2\ \beta_2\right)$ in which the allowed values are detected with 100% probability. In any other state $|\psi = (\alpha\ \beta)$ (in which the «observable» does not have a certain value) its measurement yields the same values $F_{1,2}$ with probabilities $w_{1,2} = |\langle\psi\rangle|^2$. Given the spectrum $F_{1,2}$ and the corresponding states $|F_{1,2}$ of «observable», one can construct its matrix by means of the *spectral decomposition* formula

$$\hat{F} = F_1 \cdot |F_1|F_1^{\dagger} + F_2 \cdot |F_2|F_2^{\dagger}. \qquad (3)$$

Exercise 10: It is possible to build a detector to measure a «fraction of *x*-polarized light» w_x (clearly, a polarizer transmitting only photons polarized along x is such a detector). For a single photon this quantity can take the values 1 (in the state $|x$) and 0 (in the state $|y$), while for a state $|\psi = (\alpha\ \beta)$ it does not have a certain value: each time the measurement will give either 1 or 0 and these values will be obtained with probabilities $w_x = |\alpha|^2$ and $w_y = |\beta|^2 = 1 - w_x$. Construct the matrix of this «observable».

It is important to understand that a measurement «irrevocably» changes a qubit state: after the measurement we know with certainty in which specific basic state the qubit is. Let us figure out how this affects a «probability wave». For instance, if a photon in some pure state is incident on a semi-transparent mirror, which does not destroy quantum coherence (the photon remains in pure state) but transmits the photon with a 50% probability and reflects it with a 50% probability, this means that the «probability wave» splits into transmitted and reflected waves, i.e. the photon «statistically» exists on both sides of the mirror. Suppose photon detectors are installed in front and behind the mirror, then the moment at which a detector signals to an observer about photon detection the probability wave «*collapses*». If the detector behind the mirror has signaled, the probability to detect the photon in front of the mirror instantly «vanished», now the probability to detect the photon in front of the mirror is zero.

We wish you interesting and successful work at the Olympiad!

ANSWERS TO EXERCISES.

1. $\frac{12281}{46656} \approx 26,3$ %.
2. $5 \cdot e^{i\, arcsin(0,6)}$.
3. $\frac{\sqrt{3}+i}{\sqrt{2}}$.

Comments: Of course, $-\frac{\sqrt{3}+i}{\sqrt{2}}$ is also a root of the equation $z^2 = 1 + i\sqrt{3}$. In the theory of functions of a complex variable the definition of square root is usually based on the exponential form of complex number:

$$\sqrt{z} = \sqrt{|z| \cdot e^{i\varphi}} = \sqrt{|z|} \bullet e^{i\varphi/2} = \sqrt{|z|}\left[cos\left(\frac{\varphi}{2}\right) + i \cdot sin\left(\frac{\varphi}{2}\right)\right] \qquad \text{(I)}.$$

However, extending the argument of complex number beyond the interval $[0,\ 2\pi)$ and keeping Euler's formula, one obtains the second value of the root since $e^{i(\varphi+2\pi)} = e^{i\varphi}$ and $e^{i\pi} = -1$:

$$\sqrt{z} = \sqrt{|z| \bullet e^{i(\varphi+2\pi)}} = e^{i\pi}\sqrt{|z| \cdot e^{i\varphi}} = -\sqrt{|z|}\left[cos\left(\frac{\varphi}{2}\right) + i \bullet sin\left(\frac{\varphi}{2}\right)\right].$$

By convention, (I) is called the *principal root*.

4. $(10\ 3i\ 0\ 11\ )$.
5. 6.
6. $\frac{5}{18} \approx 27,8$ %.
7. Only 2.
8. $\frac{10+4\sqrt{2}}{27} \approx 58$ %.
9. 0,57.
10. According to (3) the matrix of this observable is

$$\hat{W}_x = 1 \cdot |x|x^\dagger + 0 \cdot |y|y^\dagger = 1 \cdot (1\ 0\ )(1\ 0\ ) + 0 \cdot (0\ 1\ )(0\ 1\ ) = (1\ 0\ 0\ 0\ ).$$